Chapter 19

Runge's Theorem.

Approximation of analytic functions by rational functions.

Recall: A rational function is a quotient of polynomials.
\[ R(z) = \frac{P(z)}{Q(z)}, \quad P(z) = \sum_{n=0}^{\infty} a_n z^n, \quad Q(z) = \sum_{n=1}^{\infty} b_n z^n. \]

- \( R(z) \) is analytic except for poles at the \( \deg(Q) \) zeros of \( Q(z) \). We say that \( R(z) \) has a pole at \( z = \infty \) if \( \lim_{z \to \infty} |R(z)| = \infty \) \( \iff \deg(P) > \deg(Q) \).

- A polynomial \( p(z) \) is a rational function with poles only at \( z = \infty \).

**Ex 1**: \( G = B(0, r), \ f \in H(G) \Rightarrow P(z) = \sum_{n=0}^{\infty} a_n z^n, \ a_n = \frac{p^{(n)}(0)}{n!} \)

w/ uniform conv. on every \( K = B(0, r^*), \ r^* < r. \)

i.e. \( P_k(z) = \sum_{n=0}^{\infty} a_n z^n \) \( \to \) uniform on every compact \( K \subseteq G. \) Thus, \( f \in H(G) \) is uniform limit of polynomials.

**Ex 2**: \( G = B(0, r) \setminus \{0\}, \ f \in H(G). \Rightarrow f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \ a_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z) dz}{z^{n+1}} \) (Cauchy sum).

w/ uniform conv. on compacts \( K \subseteq G. \) Thus, \( f \) is uniform limit on compacts of rational functions.

**Rem**: If \( \{p_n\} \) seq. of polynomials, and \( \{p_n\} \) conv. uniform on any compact \( K \subseteq B(0, r) \setminus \{0\}, \) then \( \lim_{n \to \infty} p_n(z) \to f(z) \) uniform on every
compact $K \subseteq B(0, r)$. Thus, $P \to f$ in $H(B(0, r))$. In particular, $f(0) = \frac{1}{2}$, which is anal. in $B(0, r) \setminus \partial B(0, r)$ is not unif. lim. of polynomials. You need rational funs. w/ poles at $0 \& \infty$.

R ranging, then, let $K \subseteq \mathbb{C}$ be compact, and $E \subseteq \mathbb{C} \setminus K$ s.t. $E$ contains at least one point from each component of $\mathbb{C} \setminus K$. Then for every $f(z)$ analytic in $\mathbb{H}$ of $K$, there is a seq. of rational functions $\{R_n\}_{n=1}^{\infty}$ w/ poles only in $E$ s.t. $R_n \to f$ unif. on $K$.

\begin{aligned}
\text{E collection of points marked } x. \\
\end{aligned}

Prop. 1. Let $K$ be compact subset of open set $G \subseteq \mathbb{C}$. Then $f(z) = \sum_{k=1}^{\infty} \frac{1}{2\pi i} \oint_{C_k} \frac{f(\zeta) d\zeta}{z - \zeta}$.

Note $d(K, \partial G) > 0$, since $K \subseteq G$ compact. Pick $0 < \delta < \frac{1}{2} d(K, \partial G)$. Consider grid of horizontal/vertical lines advanced by $\delta$, and dense by $\delta$, s.t. $\exists$ m, the rectangles formed that intersect $K$. 

$x$ (x) First, pick $K \subseteq G$ compact s.t. $K \subseteq \text{int}(K')$. 

Note $d(K, \partial G) > 0$, since $K \subseteq G$ compact. Pick $0 < \delta < \frac{1}{2} d(K, \partial G)$. Consider grid of horizontal/vertical lines advanced by $\delta$, and dense by $\delta$, s.t. $\exists$ m, the rectangles formed that intersect $K'$.
If we pick \( z \in K \setminus \bigcup_{j=1}^{m} \partial D_j \), then \( z \in \partial D_j \) for some \( j \in \{1, \ldots, m\} \).

Then for any \( f \in \mathcal{H}(\mathbb{C}) \),

\[
\Phi(z) = \frac{1}{2\pi i} \sum_{j=1}^{m} \frac{f(z_j) d\gamma_j}{z - z_j}
\]

also \( 0 = \frac{1}{2\pi i} \int_{\partial D_j} \frac{f(z) d\gamma_j}{z - z_j} \) for all other \( j \in \{1, \ldots, m\} \setminus \{j_0\} \).

Thus

\[
\Phi(z) = \frac{1}{2\pi i} \sum_{j=1}^{m} \frac{f(z_j) d\gamma_j}{z - z_j}
\]

The boundaries \( \partial D_j \) of boxes \( D_j \), \( j = 1, \ldots, m \), form a finite \( P \) of closed paths \( \Gamma_k \), \( k = 1, \ldots, l \), enclosing \( \bigcup_{j=1}^{m} D_j \), where we delete a line segment if it intersects \( K = \Sigma \Gamma \) twice with opposite signs in the path \( \bigcup_{j=1}^{m} \partial D_j \).

If \( \gamma \) is such a line segment, then the contribution cancels in \( \Phi(z) \).

\[
\Phi(z) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\partial D_j} \frac{f(z) d\gamma_j}{z - z_j} = \frac{1}{2\pi i} \int_{\bigcup_{j=1}^{m} \partial D_j} \frac{f(z) d\gamma_j}{z - z_j}
\]
\[ \sum_{j=1}^{\infty} \frac{1}{2\pi i} \int_{C_{1j}} \frac{f(z)dz}{z - z} \]

Now, if \( z \in \bigcup_{j=1}^{\infty} \Omega_{1j} \cap K \), then \( z \) must lie on a deleted line segment, since \( z \in \text{int} \ K \) so \( z \) must be on \( \partial \Omega_{1j} \) for two/four adjacent boxes. By continuity, we get

\[ f(z) = \sum_{j=1}^{\infty} \frac{1}{2\pi i} \int_{C_{1j}} \frac{f(z)dz}{z - z} \]

[\Box]