Lecture 7

**Arzela–Ascoli Thm.** A family \( F \subseteq C(G, \Omega) \) is normal \( \iff \)

(i) \( \forall z \in G, \{ f(z) : f \in F \} \) is compact.

(ii) \( F \) is equicontinuous.

**Pr:** \( \Rightarrow \): See Lecture 6 notes.

\[ \Leftarrow \]. We shall need the following countable version of Tychonoff's Thm.

**Prop 1.** Let \( \{ (X_n, d_n) \}_{n=1}^{\infty} \) be sequence of metric spaces, and set

\[ X = \bigcap_{n=1}^{\infty} X_n, \quad d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} \quad \forall x = \{ x_n \}, y = \{ y_n \} \in X. \]

(a) \( \varphi_k \to \varphi \iff \forall n, \lim_{k \to \infty} x_n^k = x_n \)

(b) If each \( (X_n, d_n) \) is compact, then \( (X, d) \) is compact.

**Pr:** See Conway (VII.1.119 Prop).

**Pr of \( \Leftarrow \) in AAThm:** Let \( \{ z_n \} \) be sequence of rational (Re and Imp part) points in \( G \), and set \( X_n = \{ f(z_n) : f \in F \} \subseteq \Omega \), w/ metric \( d = d_G \) by (i), each \( (X_n, d_n) \) is compact. Set \( X = \bigcap_{n=1}^{\infty} X_n \)

w/ metric \( d = d_X \) as in Prop 1 \( \Rightarrow (X, d) \) is compact.

Let \( \{ f_k \} \) be sequence in \( F \). Since \( C(G, \Omega) \) is complete,

suffices to show that \( \{ f_k \} \) has subseq. \( \{ f_{k_j} \} \)

that is Cauchy. Let \( \varphi_k = \{ f_k(z_n) : n \in \mathbb{N} \} \in X \).

Since \( X \) compact \( \Rightarrow \) \( \exists \) subsequence \( \{ f_{k_j}(z_n) \}_{n \in \mathbb{N}} \)

that converges \( \varphi_{k_j} \to \varphi_0 \) in \( X \).

**Claim:** \( \{ f_{k_j} \} \) Cauchy in \( C(G, \Omega) \). \( \Rightarrow \) \( \varphi_0 \).
Claim: \( \{f_{ij}\} \) Cauchy in \( C(G, \Omega) \). (\( \Rightarrow \) ?)

Proof of Claim. By prev. Prop., suffices to show that for any \( \varepsilon > 0 \) and \( \bar{K} \leq K \) compact, \( \exists I \) s.t. \( d_1(p_{k}, p_{k}^*) < \varepsilon, i,j \in I \). We shall replace \( K \) by larger \( K' \leq \bar{K} \) compact as follows:

Let \( R = d(K, C(G)) > 0 \Rightarrow K' = \{ z \in G : d(z, K) \leq R/2 \} \leq \bar{K} \) compact.

If equivalent at every \( z \in G \Rightarrow f \) is uniformly equivalent on compact \( K' \); i.e. \( \exists \delta > 0 \) s.t. \( z, w \in K', |z - w| < \delta \Rightarrow d(f(z), f(w)) < \varepsilon/3, \forall f \in F \). (\( \Rightarrow \) Lebesgue Covering Lemma; see Conway).

Now \( \{ B(z_n, \delta) : z_n \in K' \} \) is open covering of \( K \Rightarrow \exists \)

finite subcover, say, \( K \leq \bigcup_{n=1}^{m} B(z_n, \delta) \) (remembering \( \{ z_n \} \) )

Pick \( I \) s.t. \( d(p_{k_i}(z_n), p_{k_j}(z_n)) < \varepsilon/3, \forall n=1, \ldots, m, \) if \( i,j \in I \).

Now, for any \( z \in K \), \( z \in B(z_n, \delta) \) for some \( n=1, \ldots, m \), and

\[
d(f_{k_i}(z), f_{k_j}(z)) \leq d(f_{k_i}(z), f_{k_i}(z_n)) + d(f_{k_i}(z_n), f_{k_j}(z_n)) + d(f_{k_j}(z_n), f_{k_j}(z))
\]

\[
< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\]

\( \Rightarrow d(p_{k_i}, p_{k_j}) < \varepsilon \), as desired. \( \Box \)