Recall: The Poisson kernel $P_r(\theta)$, $0 < r < 1$, $\theta \in \mathbb{R}$ (2π-periodic).

Thus, $\forall u \in C(\bar{\Omega})$ and unique sol. $u \in C(\bar{\Omega}) \cap C^\infty(\Omega)$ s.t.

\[
\begin{cases}
\Delta u = 0 \text{ in } \Omega \\
u|_{\partial \Omega} = u_0.
\end{cases}
\]  
(DP)

Moreover,

\[
u(r e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) u_0(e^{it}) \, dt.
\]

Proof. See Lecture 10 notes.

Useful to deduce P-kernel in general disk $B(a, R)$, let $u$ solve (DP), and consider $\tilde{u}(z) = u\left(\frac{z - a}{R}\right)$. Then $\tilde{u} \in C(\bar{B}) \cap C^\infty(B)$, $\Delta \tilde{u} = 0$ in $B$, and $\tilde{u}|_{\partial B} = \tilde{u}_0$, $\tilde{u}_0(z) = u_0\left(\frac{z - a}{R}\right)$, so $\tilde{u}$ solves DP in $B$ w/ data $\tilde{u}_0$ on $\partial B$.

\[
\tilde{u}(z + r e^{i\theta}) = u\left(\frac{z + r e^{i\theta} - a}{R}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_{r R}(\theta - t) \tilde{u}_0(e^{it}) \, dt
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - (r/R)^2}{1 + (r/R)^2 - 2r/R \cos(\theta - t)} \tilde{u}_0(a + re^{it}) \, dt
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - r^2 - 2rR \cos(\theta - t)} \tilde{u}_0(a + re^{it}) \, dt
\]

\[
\tilde{u}(\theta - t) = \frac{q(x)^2}{|Re^{i\theta} - Re^{i\theta}|^2}
\]
Two Corollaries:

Cor 1. If $u$ has MVP in $G$, then $u$ is harmonic in $G$.

Proof: Let $\bar{B} = \overline{B(a, r)} \subseteq G$, and let $u$ be sol. to MP w/ $\bar{u}_{\partial} = u|_{\partial \bar{G}}$. Then, $\bar{u}$ is harm. in $B$, is $C(\overline{B})$, and has MVP in $B$. Moreover $\bar{u} - u = 0$ on $\partial B \iff \bar{u} - u = 0$ in $B$ by MP-corollary. $\Rightarrow u$ is harm. in $B$. $\Box$

Cor 2. (Harnack's Inequality) Suppose $u$ is harm. in $B = B(a, R)$, cont. in $\overline{B}$, and $u \geq 0$. Then, $\forall 0 \leq r \leq R$:

$$\frac{R-r}{R+r} u(a) \leq u(a + re^{i\theta}) \leq \frac{R+r}{R-r} u(a).$$

Proof: $P_r(\theta - \xi) = \frac{R^2 - r^2}{|Re^{i\theta} - re^{i\xi}|^2}$ and:

$$R - r \leq |Re^{i\theta} - re^{i\xi}| \leq R + r. \quad (x) \Rightarrow \Box$$