The Dirichlet Problem.

Let \( G \subseteq \mathbb{C} \) be a region and consider the Dirichlet problem for \( f \in C(\partial G, \mathbb{R}) \):

\[
\begin{aligned}
\Delta u &= 0 \quad \text{in} \quad G \\
\frac{\partial u}{\partial n} &= f \quad \text{on} \quad \partial G
\end{aligned}
\]

\[ u|_{\partial G} = 0 \quad \text{boundary value} \]

Def 1. \( G \) is said to be a Dirichlet region if (1) has a (unique) solution for every \( f \in C(\partial G, \mathbb{R}) \). [Recall Perron's method: \( u(z) := \sup \{ \varphi(z) : \varphi \in \Phi \} \]}

Obs. Not all regions are D-regions.

Ex. \( G = \mathbb{D} \setminus \{0\} \). Consider DP w/ BV \[ \Phi(z) := \begin{cases} 0 & z \in \partial \mathbb{D} \\ 1 & z = 0 \end{cases} \]

Prop. A solution \( u \) to DP w/ BV \( \Phi \).

If. Suppose \( u \) is a sol. and consider \( u_\mathbb{D}(z) := \frac{1}{\log \mathbb{D}} \) in \( G_\mathbb{D} := \overline{G \setminus \mathbb{D}(0,1)} \). Note that \( u_\mathbb{D}(z) = 0 \) on \( \partial \mathbb{D} \) and 1 on \( \partial \mathbb{D}(0,1) \).

Since \( 0 \leq \mathbb{D} \subseteq G \leq G_\mathbb{D} \) and \( \lim_{z \to \partial \mathbb{D}} u(z) = 0 = u_\mathbb{D}(z) \Rightarrow \) Max. Princ. \( \Rightarrow 

\begin{align*}
&u \leq u_\mathbb{D} \quad \text{in} \quad G_\mathbb{D}, \quad \forall \epsilon > 0. \quad \text{Pick any} \quad z_0 \in \mathbb{D} \quad \text{and note that} \\
&u(z_0) \to 0 \quad \text{as} \quad \epsilon \to 0 \Rightarrow u(z_0) = 0 \Rightarrow u = 0 \quad \text{in} \quad G.
\end{align*}

But then \( \lim_{z \to 0} u(z) = 0 \neq 1 \). \( \Box \)

We may characterize D-regions by a local potential theoretic conditions near \( \partial G \).

Def 2. Let \( G \) be region. A barrier at \( a \in \partial G \) is a family \( \{ U_r \} \) of superharmonic fans in \( G(a,r) := G \cap B(a,r) \) s.t. \( 0 \leq U_r \leq 1 \) &

\[
\lim_{r \to a} U_r(z) := \begin{cases} 0 & z = a \\
1, & z \in \partial B(a,r) \cap G
\end{cases}
\]
For convenience, we may extend each \( \psi_r \) to superharmonic \( \psi \) in \( G \) by setting it to be \( \psi=1 \) in \( G \setminus S(a,r) \).

**Theorem 1.** Let \( G \) be region. \( \psi \) is:

(i) \( G \) is a region.
(ii) \( G \) has a barrier at \( a \) for every \( a \in \partial G \).

**Proof.** (i) \( \Rightarrow \) (ii). Pick \( a \in \partial G \). Assume first \( a \neq \infty \) and consider BV
\[
\psi(z) = \begin{cases} \frac{|z-a|}{1+|z-a|}, & z \neq a \\ 1, & z = \infty \end{cases}
\]
Then \( \psi \in C(\partial G, \mathbb{R}) \) and we may

Solve \( DP \) w/ BV \( \psi \), solution \( u \). Note that \( 0 < u < 1 \) in \( G \setminus \{a\} \) by Max. Princ. (\( u \) not constant), so we may define \( \bar{G} = \{ w \in \mathbb{C} : \lim_{z \to w} u(z) > 0 \} \).

The function \( \psi_r(z) = \frac{1}{C_r} \min \{ u(z), C_r \} \), \( z \in G(a,r) \), is a barrier at \( a \in \partial G \). For \( a = \infty \), use \( z \to \frac{1}{z} \) to send \( \infty \) to 0.

(\( z \to \frac{1}{z} \) preserves subharmonic superharmonic functions.

(ii) \( \Rightarrow \) (i). Pick \( a \in \partial G \). \( \psi \in C(\partial G, \mathbb{R}) \): \( a \neq \infty \) and \( \psi(a) = 0 \). Pick \( \varepsilon > 0 \).

Let \( \bar{G}_r \) be barrier at \( a \) (extended by 1 outside \( G(a,r) \)).

Let \( M := \sup_{z \in \partial G} \psi \) and consider \( \Phi_r := \psi_r + \varepsilon \). For \( r \) sufficiently small, \( \Phi_r \) will be in \( PC(\bar{G}, \mathbb{R}) \) (see Conway), and similarly
\( \Phi_r = M \psi_r + \varepsilon \) is superharmonic and \( \psi \leq \Phi_r \) for every \( \psi \in PC(\bar{G}) \).

\( \Rightarrow \) \( \phi_r \leq u \leq \Phi_r \Rightarrow \lim_{z \to a} u(z) = a \ldots \)

Details next lecture...