Recall that for a region $G \subseteq \mathbb{C}$, a barrier at $a \in \partial G$ is a family $\{\psi_r\}$ of superharmonic functions $0 \leq \psi_r \leq 1$ in $G(a, r) = G \cap B(a, r)$ such that $\lim \psi_r(z) = 0$, $\lim \psi_r(z) = 1$ as $z \to a$, $z \to b$ respectively.

**Theorem 1.**

(a) $G$ is a D-region (no到底 for all $f \in C(\partial G)$).
(b) $G$ has barrier at each $a \in \partial G$.

**Proof.** (a) $w = \sup \{\psi_r \in \mathcal{P}(G) \mid 0 \leq \psi_r \leq 1\}$ is a Perron function. WLOG $a \neq b$, $f(a) = 0$.

Need to verify $\lim \psi_r(z) = f(a)$. Have $|f(z)| \leq M$ on $\partial G$. Fix $\varepsilon > 0$ and set $\xi = \{M, \varepsilon, + \}$, $\zeta = \{M, \varepsilon, -\}$; sub- and superharmonic resp. in $G$.

For $r > 0$ sufficiently small, $|f(z)| \leq \xi$ on $\partial B(a, r) \cap \partial G$.

Fix such $r = \delta$.

**Claim 1.** $\xi \in \mathcal{P}(G, \delta)$.

**Proof.** If $z \in \partial B(a, r) \cap \partial G$, $\xi(z) \leq -\xi \leq f(b)$ by choice of $\delta = \delta > 0$.

Thus, $\limsup \xi(z) = f(b) = \Phi \xi(z) = f(b) = \Phi \xi(z) = f(b)$. All

• But then $\Phi \xi(z) = \Phi \xi(z) \Rightarrow -\delta = \liminf \Phi \xi(z) \leq \limsup \Phi \xi(z) = \Phi \xi(z)$. (1)

**Claim 2.** $\Phi \xi \geq \Phi$, $\forall \Phi \in \mathcal{P}(G, d)$.

**Proof.** We check $\limsup \Phi(z) \leq \Phi \xi(z)$, $\forall \Phi \in \mathcal{P}$, $b \in \partial G$.

But $\limsup \Phi(z) \leq f(b)$, so we check $\liminf \Phi \xi(z) \geq f(b)$, which follows from Claim 1 since $\Phi \xi = \Phi \xi$. All

• Then $\liminf \Phi \xi(z) \leq \liminf \Phi \xi(z) \Rightarrow \limsup \Phi \xi(z) \leq \Phi \xi(z)$. (2)

(1) + (2) $+ \varepsilon > 0$ and $\Rightarrow \lim \Phi \xi(z) = f(a)$. All
Theorem. Let $G \subset \mathbb{C}$ be region. If every $a \in \partial G$ is such that the component of $\mathbb{C} \setminus G$ that contains $a$ is not $\{a\}$, then $G$ is $D$-region.

Sketch of Proof. Check that $G$ has barrier at every $a \in \partial G$. Pick $a \in \partial G$.

Step 1: Set $G_0 = G$ the complement of $G$. Then $G_0$ is simply connected (verify).

Then $\exists$ branch $l(z) = \log z$ in $\mathbb{C} \setminus \mathbb{D}$. Consider conformal map

$$ f_r(z) = -l(z) + \log r \quad \text{in} \quad G_r(a=0; r) , \quad \text{Re} f_r(z) = \log \frac{r}{2} > 0 \quad \text{in} \quad G_r. $$

Let $I_k = (\alpha_k, i\beta_k)$, $k=1,2,\ldots$ where $\alpha_k, \beta_k$ determined by branch of log. Nevertheless, have (**) $\sum_{k=1}^{m} (\beta_k - \alpha_k) \leq 2\pi$.

Next picture:

$$ w = \frac{z - i\alpha_k}{z - i\beta_k} \rightarrow \text{Log } w $$

$\text{Re } z > 0$
Consider \( h_n(z) = \frac{1}{\pi} \text{Im} \log \left( \frac{z+i\alpha_n}{z-1/\alpha_n} \right), \quad 0 < h_n(z) < 1 + \text{harmonic} \).

Claims:

1. \[ h_n(z) = \frac{1}{\pi} \int \frac{x dt}{\sqrt{x^2+(y-t)^2}} \]

2. \( h_1(z) + \ldots + h_k(z) \) harmonic, \( h(z) = \sum_{k=1}^{\infty} h_k(z) \), \[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-|x|} dt}{\pi x} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{ds}{1+s^2} = 1. \]

3. \( h(x+iy) \leq \frac{1}{\pi x} \sum_{k=1}^{\infty} \frac{1}{\alpha_k} \) \( \text{by (8)} \)

4. \( \lim_{z \to iy} h_n(z) = \begin{cases} 0 & y \notin [\alpha_1, \rho] \\ 1 & y \in [\alpha_1, \rho] \end{cases} \) \( \alpha_k \)

A careful analysis (see Conway) also shows

\[ \lim_{z \to iy} h(z) = \begin{cases} 0 & y \notin [\alpha_1, \rho] \\ 1 & x \in U(\alpha_1, \rho) \end{cases} \]
\[
\begin{align*}
\lim_{z \to 1^+} h(z) = 1, & \quad z \in \Omega_+(\theta_0, \rho_0), \\
\lim_{z \to \infty} h(z) = 0, & \quad \Re z > 0,
\end{align*}
\]

Now, \( \varphi_r(z) = (h \circ f_r)(z) \) is harmonic in \( G_r \), \( 0 \leq \varphi_r \leq 1 \),

\[
\begin{align*}
\lim_{z \to 0} \varphi_r(z) = \lim_{z \to \infty} h(z) = 0, & \quad \lim_{z \to b} \varphi_r(z) = \lim_{z \to i \infty} h(z) = 1, \\
& \quad |b-a|=r, \quad \Re z > 0, \quad z \in \Omega_+(\theta_0, \rho_0).
\end{align*}
\]

\[
\Rightarrow
\]

\( \{\varphi_r\} \) is a barrier at \( a \).