Recall. A **complex manifold** of dim 1 is a connected Hausdorff space $X$ w/ atlas $\Phi$ of charts $\{(U_x, \varphi_x)\}$:

$$
\xymatrix{
U_x \ar[rd]_{\varphi_x} & & X \ar[ld]^\varphi \\
& \mathbb{C} & 
}
$$

**Transition functions** $\varphi_y \circ \varphi_x^{-1}$ are analytic.

**Def.** If $X, Y$ are complex manifolds, $f: X \rightarrow Y$ cont., then $f$ is analytic if $\forall x \in X \exists$ chart $(U, \varphi_x), (V, \varphi_y)$ on $X, Y$ respectively, s.t. 

$x \in U$, $f(x) \in V$ and $\varphi_y \circ f \circ \varphi_x^{-1} : \varphi_x(U) \rightarrow \varphi_y(V)$ analytic.

**Prop.** Notion is independent of chart chosen.

$$
\xymatrix{
X \ar[r]^-f & Y \\
U_x \ar[r]^-{\varphi_x} & \mathbb{C} \\

U_x \ar[r]^-{\varphi_x} & \mathbb{C} \\
& \mathbb{C} & 
}
$$

$$
\alpha = \varphi_y \circ f \circ \varphi_x^{-1} \\
\beta = \psi_y \circ f \circ \psi_x^{-1}
$$
\[ g_x : \psi_x \circ \varphi^{-1} \Rightarrow h = \psi_x^{-1} \circ g_x \circ \varphi_x = \]

\[ g_x : \psi_x \circ \varphi^{-1} \circ g_x \circ \varphi_x \Rightarrow \text{anal.} \]

\[ \text{anal.} \Rightarrow g_x \text{ anal.} \]

**Remark:** A differentiable manifold of dimension \( n \) can be defined by coordinate charts \( \{ (U_i, \psi_i) \} \), \( \psi_i : U_i \to \psi_i(U_i) \in \mathbb{R}^n \) and "analytic" replaced by "differentiable".

2. Two different atlases \( \Phi, \Phi' \) on \( X \) give same analytic structure if \( \Phi \cup \Phi' \) is an atlas, i.e., their charts are compatible. For example, given \( \Phi = \{ (U, \psi) \} \), you can create \( \Phi' \) by adding chart \( (V, \psi) \), where \( V \subseteq U \) and \( \psi : \psi(U) \to \mathbb{R}^n \) is "shrinkable".

Many important results in complex analysis carry over to the setting of complex manifolds:

1. If \( f, g : X \to \mathbb{C} \) analytic, \( Z : = \{ x \in X : f(x) = g(x) \} \) has limit point in \( X \), then \( f = g \). (Uniqueness).

   **Proof:** Let \( \{ x_n \} \) be seq. in \( Z \) and \( x_n \to x_0 \) in \( X \). Let \( (U, \varphi) \) be a chart at \( x_0 \), \( (V, \psi) \) chart at \( y_0 = \varphi(x_0) = g(x_0) \). For \( n \geq N \), \( x_n \in U \) and \( y_n = \varphi(x_n) = g(x_n) \in V \). Consider analytic functions \( \tilde{f}(z) = (\varphi \circ f \circ \varphi^{-1})(z) \), \( \tilde{g}(z) = (\varphi \circ g \circ \varphi^{-1})(z) \). They coincide on \( Z \cap \varphi(U) \), \( z_n \to z_0 \in \varphi(U) \Rightarrow \tilde{f} = \tilde{g} \text{ in } \varphi(U) \) by standard uniqueness \( \Rightarrow \tilde{f} = \tilde{g} \text{ in } U \).

   Next, let \( Z' : = \{ x \in X : \exists \text{ open } W \subseteq X, x \in W, f = g \text{ on } W \} \). \( Z' \) is
open by def, nonempty by above. If \( x_0 \) is a limit point of \( Z \), then argument above also shows \( x_0 \in Z' \). Thus, \( Z' \) nonempty open and closed \( \Rightarrow Z' = X \) by connectedness. \( \Box \)

\[ (2) \quad \text{If } f : X \to \mathbb{C} \text{ is analytic, } \exists x_0 \in X \text{ s.t. } |f(x_0)| = \sup_{x \in X} |f(x)| \text{ then } f \text{ is constant. (Max Mod Principle).} \]

\[ (3) \quad \text{If } X \text{ compact, } f : X \to \mathbb{C} \text{ analytic, then } f \text{ is constant. (Liouville)} \]

\[ (4) \quad \text{If } f : X \to \mathbb{C} \text{ is analytic, } U \subseteq X \text{ open, then } f(U) \subseteq \mathbb{C} \text{ is open (open mapping theorem).} \]