Lecture 12
Friday, April 26, 2019 5:53 AM

From last time: If \( f(z) \) has finite rank \( P \) and genus \( \mu \), then we can write \( f(z) \) in non-standard form \( f(z) = z^\mu e^{\tilde{g}(z)} P(z) \) using \( \mu \) instead of \( P \), i.e. \( \tilde{P}(z) = \prod_{n=1}^\infty E_n(2^\lambda n) \). Then, \( \text{deg} \tilde{g} \leq \mu \). So in proof from last lecture, we may indeed proceed assuming \( P=\mu \). (See also from Lecture 11 notes.)

Recall, \( f(z) \) entire w/ zero seq. (not incl. \( 0 \)) \( \{a_n\}_n \).

1. Finite rank \( P \iff \sum_{n=1}^\infty \frac{1}{|a_n|} < \infty \iff f(z) = z^\mu e^{g(z)} P(z) \)
2. Finite genus \( g \) poly and \( \mu = \max(P, \text{deg } g) \).
3. Finite order \( \lambda \iff \lambda = \inf \{ a > 0 : |P(z)| \leq e^{1/a}, |z| > R \} \).

Shown: \( \lambda \leq \mu + 1 \). Now, want to show \( \mu \leq \lambda \).

Need prelims. \( M(r) = \sup \{|f(z)| : |z| = r\} \)

Lemma 1. \( f(z) \) is of finite order \( \lambda \iff \lim_{r \to \infty} \frac{\log \log M(r)}{\log r} = \lambda \)

Pf: See Lecture 11 notes. \( \square \)

Set \( n(r) = \# \{ \text{zeros of } f \text{ in } B(0, r) \} = \sup \{|k : a_k, \ldots, a_n \in B(0, r)\} \) if \( |a_k| \leq |a_{k+1}| \ldots \)

Lemma 2. Assume \( f(0) = 1 \), then \( n(r) \log 2 \leq \log M(2r) \)

Pf: Jensen's formula at \( z = 0 \) in \( B(0, 2r) \)

\[
0 = \log |f(0)| = -\sum_{n=1}^{n(2r)} \log \frac{2r}{|a_n|} + \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta
\]

\[
< -\sum_{n=1}^{n(2r)} \log \frac{2r}{|a_n|} + \log M(2r)
\]

\[
\sum_{n=1}^{n(2r)} \log \frac{2r}{|a_n|} \leq \log M(2r). \quad \text{If } |a_n| < r \quad \Rightarrow \frac{2r}{|a_n|} > 2
\]

\[
\Rightarrow n(r) \log 2 \leq \sum_{n=1}^{n(r)} \log \frac{2r}{|a_n|} \leq \log M(2r). \quad \square
\]

Hadamard’s Factorization Theorem. If \( f(z) \) entire of finite order \( \lambda \), then
Hadamard's Factorization Theorem. If \( f(z) \) entire of finite order \( \lambda \), then 
\( f(z) \) has finite genus \( \mu \leq \lambda \). \( (\Rightarrow p \leq \lambda) \)

Remark. HFT says that a growth condition on \( |f(z)| \) forces growth of zero seq. \( f_{a,n} \).

\[ \text{Proof.} \]

1. Show \( f(z) \) has finite rank \( p \leq q = 1 + \lambda \).

\[ \text{Let } q = 1 + \lambda \text{ (integer with } q \leq \lambda < q + 1). \]

Must show \( \sum_{k=1}^{q+1} \left( \frac{1}{\lambda^k} \right) < \infty \).

WLOG, \( f(0) = 1 \), b/c if \( f(z) = cz^m \tilde{f}(z) \) w/ \( \tilde{f}(0) = 1 \), then \( f(z) \) is of finite order \( \lambda \) as well. \( (|z|^m \leq e^{\mu z}, |\tilde{f}| \leq e^{\mu z}, \forall z > 0) \).

Thus, \( \tilde{f}(z) \) is of finite order \( \lambda \) as well. \( (|z|^m \leq e^{\mu z}, |\tilde{f}| \leq e^{\mu z}, \forall z > 0) \).

By Jensen Cor., \( k \leq \mu \left( \frac{1}{\lambda^{k+1}} \right) \leq \frac{\log M(2\lambda^k)}{\log 2} \leq \frac{1}{\lambda^k} \log 2 \), \( \forall z > 0 \) and \( (\lambda(1+\epsilon)/\lambda) \),

\[ i.e., \lambda^k + \epsilon \geq \frac{C(\epsilon)}{\lambda} \Rightarrow \lambda^k \geq C(\epsilon) \text{ for } k \gg 1. \]

Since \( q+1 > \lambda \), choose \( \alpha > 1 \) s.t. \( q+1 > \alpha \lambda \) and \( \epsilon > 0 \) s.t. \( q+1 > \alpha(1+\epsilon)/\lambda \), \( \forall \epsilon > 0 \). Then \( \lambda^k \geq C(\epsilon) \) \( \text{ for } k \gg 1. \)

\[ \sum_{k=1}^{\infty} \left( \frac{1}{\lambda^k} \right) \text{ is majorized by } \sum_{k=1}^{\infty} \frac{1}{\lambda^k} < \infty \text{ since } \alpha > 1. \]

2. Standard form \( f(z) = e^{g(z)} p(z) \), \( p(z) = \prod_{n=1}^{\infty} E_p(\frac{3a_n}{n}) \), \( p \leq \lambda \).

Must show \( g(z) \) is poly of deg \( \leq \lambda \).

Have \( \log f = g + \log p \Rightarrow \frac{f'}{f} = g' + \frac{p'}{p} (\times) \)

Show: \( (\frac{d}{dz}) g' = 0 \) \( (\Rightarrow g \text{ poly of deg } \leq q - 1 \Rightarrow g \text{ poly of deg } \leq q \Rightarrow \Theta) \)

\[ \text{suffices to check for } z \in B = B(\alpha, \epsilon) \text{ s.t. } a_n \leq \alpha \epsilon. \]

\[ \frac{(d)}{dz} p' = (\frac{d}{dz})^{q+1} \log p = \frac{d}{dz} \sum_{n=1}^{\infty} \log E_p(\frac{3a_n}{n}) = \sum_{n=1}^{\infty} \left( \frac{d}{dz} \right)^{q+1} \log E_p(\frac{3a_n}{n}). \]
\[
\begin{align*}
\frac{\partial}{\partial z} \log P &= \frac{q_{\text{ret}}}{\partial E_{\text{ret}}(z)} \log P = \frac{q_{\text{ret}}}{\partial E_{\text{ret}}(z)} \log E_{\text{ret}}(z) \\
&= \frac{\partial}{\partial z} \log E_{\text{ret}}(z) = \frac{1}{a_n} \frac{E_{\text{ret}}(z/a_n)}{E_{\text{ret}}(z)} \cdot E_{\text{ret}}'(z) = - \frac{z^{2+\cdots+p}}{2^p} + \left(1+\cdots+z^p\right)(1-z^2)
\end{align*}
\]

\[
E_{\text{ret}}'(z) = \left(1-2^p\right)(1-z^2)^{p+1}
\]

\[
\Rightarrow \frac{E_{\text{ret}}'(z)}{E_{\text{ret}}(z)} = - \frac{2^p}{1-2} = \frac{1-2^p-1}{1-2} = 1 + 2 + \cdots + 2^p - \frac{1}{1-2}
\]

\[
(d^q \frac{\partial}{\partial z}) \log E_{\text{ret}}(z/a_n) = \frac{1}{a_n^{q+1}} \left(\frac{d}{dz}\right)^{q+1} \log E_{\text{ret}}(z/a_n) = - q! \left(\frac{q}{a_n-2}\right)^{q+1}
\]

\[
(d^q \frac{\partial}{\partial z}) \log P(z) = - q! \sum_{n=1}^{\infty} \frac{1}{(a_n-2)^{q+1}} \quad (q \leq q', \quad q' = \lambda L).
\]

**Prop.** \((d^q \frac{\partial}{\partial z}) q^s = 0 \Rightarrow \square.

**Pf.** of **Prop.** Next lecture.

**Corollaries.**

1. If \(f(z)\) has finite order, then \(f(z)\) assumes every value \(z \in C\) with at most one exception. \((f(z) = \infty \text{ never } o\), but taken every other value\).

**Pf.** Assume \(f(z) \neq w_1\) and \(f(z) \neq w_2\) w/ \(w_1 \neq w_2\). Then, \(f(z)-w_1 = e^{g(z)}\) and \(e^{g(z)}\) has finite order. By Hadamard \(g(z)\) is a polynomial. Since \(f(z) \neq w_2 \Rightarrow e^{g(z)} = f(z)-w_1 \neq w_2 - w_1 \neq 0\), but \(g(z)\) is polynomial.
\( e^g(z) = f(z) - w_1 \neq w_2 - w_1 \neq 0 \); but \( g(z) \) is polynomial (which takes every value including \( \log(w_2 - w_1) \)) which is a contradiction. \( \square \)

(2) If \( f(z) \) has finite order \( \lambda, \lambda \notin \mathbb{Z} \), then \( f(z) \) has infinitely many zeros.

**Pf.** Show finite # zeros \( a_1, \ldots, a_n \Rightarrow \lambda \in \mathbb{Z} \). Let \( h(z) \) be polynomial w/ same zeros. Then \( \frac{f(z)}{h(z)} \) is entire, no zeros, and same argument as in \( \lambda \notin \mathbb{Z} \) above, \( \frac{f(z)}{h(z)} \) has same finite order, \( \lambda \). But by Hadamard \( \frac{f(z)}{h(z)} = e^g(z) \) where \( g(z) \) is polynomial and \( e^g(z) \) has integral order \( \lambda = \deg g \). \( \square \)