Barriers.

Def. A barrier for \( G \subseteq C \) at \( a \in \partial C \) is a family
\[
\{ \psi_r \}_{0 < r < r_0} \text{ of super-harmonic functions in } G(a,r) = B(a,r) \setminus \partial C
\]
s.t. \( 0 \leq \psi_r \leq 1 \) and
\[
\lim_{z \to a} \psi_r(z) = 0, \quad \lim_{z \to b} \psi_r(z) = 1, \quad \forall b \in \partial B(a,r) \setminus \partial C.
\]

Rem. We extend \( \psi_r \) to superharmonic \( \bar{\psi}_r \) on \( G \) by setting \( \bar{\psi}_r = 1 \) in \( G \setminus G(a,r) \). Easy to check \( \bar{\psi}_r \) superharmonic. (Just need to check super MMP for \( \phi \approx \bar{\phi}_r \) on \( \partial B(a,r) \setminus \partial C \).

Prop. Let \( G \subseteq C \) be region.

(i) If \( \text{DP} \) is solvable in \( G \), then \( G \) has barrier at every \( a \in \partial C \).
(ii) If \( G \) has barrier at \( a \in \partial C \), then \( \lim_{z \to a} u(z) = f(z) \), where \( u(z) \) is Perron function \( u(z) = \sup \{ \phi(z) : \phi \in PC_G(f) \} \)

Pf. (i) For \( r > 0 \) sufficiently small, let \( u(z) \) be solution to \( \text{DP} \) \( \Delta u = 0 \), \( u = f \) on \( \partial G \), where \( f(a) = 0 \), \( 0 < f \leq 1 \) in \( \delta G \setminus \partial C \), and \( f(b) = 1 \) for some \( b \in \partial C \).

(If \( a \neq 0 \), then \( f(z) = \frac{|z-a|}{|z+1|} \) will do for any \( \Gamma \).) By Max Princ.,
\[
0 < u(z) < 1 \quad \text{in } G, \quad \lim_{z \to \partial C} u(z) = 0.
\]
Now, let \( c_r := \inf_{z \in \partial B(a,r)} u(z) \).
We have \( 0 < c_r < 1 \), since \( u \) extends cont. to compact set \( \overline{\partial B(a,r)} \), and \( 0 < u(z) < 1 \) on \( G \) w/ \( u(z) > 0 \) on \( \partial B(a,r) \setminus \partial C \).
Set \( \psi_r(z) := \inf_{c_r} \frac{1}{c_r} \inf_{z \in \partial B(a,r)} u(z) \) in \( G(a,r) \). Check that satisfies req's.

(ii) Let \( \{ \psi_r \}_{0 < r < r_0} \) be barrier w/ extension \( \bar{\psi}_r \) to \( G \). Let \( M := \max f \). \( \forall a \in \partial C \).

WLOG, \( f(a) = 0 \). Pick \( \delta > 0 \) and \( \delta > 0 \) s.t. \( |f(z)| < \varepsilon \) for \( |z-a| < \delta \).
Consider \( \phi_0(z) = -M \bar{\psi}_r - \varepsilon \), where \( \varepsilon < \delta/2 \).

Claim 1. \( \phi_0 \in PC_G(f) \).

Claim 2. \( \phi_0(z) < M \) \( \forall z \in \partial C \).
Claim 1. \( \phi_0 \in P(G_a, f) \).
- \( \phi_0 \) is subharmonic.
- \( \phi_0(z) \leq -M - \epsilon \) in \( G \setminus G(a, r) \) and \( f(z) \geq -M \).
- \( \phi_0(z) \leq -\epsilon \) in \( G(a, r) \) and \( f(z) \geq -\epsilon \). 

Consider superharmonic. \( y(z) = M \psi r + \epsilon \). \( \liminf_{z \to b} y(z) \geq f(b) \)
for \( b \not\in G \). By Max. Princ., if \( \phi \in P(a, f) \), then \( \phi \leq \psi_0 \) in \( G \).

Since \( u(z) \) is Perron function \( \Rightarrow -M \psi r - \epsilon \leq u \leq M \psi r + \epsilon \) in \( G \).

Taking \( \lim_{z \to a} u(z) \) \( \Rightarrow -\epsilon \leq \lim_{z \to a} u(z) \leq \epsilon \). Since \( \epsilon > 0 \) and \( f \).

\( \lim_{z \to a} u(z) = 0 = f(a) \). 

Cor 1. \( G \subseteq C \) is a **Dirichlet region** (i.e. DP solvable for all \( f \in C(DG_a) \)). 

\[ \Rightarrow \]

\( G \) has barrier at every \( a \in \partial G \).