MATH 220A
Midterm Exam, November 8, 2019

Instructions: 3 hours. You may use without proof results proved in Conway up to and including Chapter IV. When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail.

1. (15+15=30p) Suppose that the radius of convergence (ROC) of the power series $\sum_{n=0}^{\infty} a_n z^n$ is $R > 1$.

   (a) Show that $\lim_{n \to \infty} a_n = 0$.

   (b) Compute ROC for $\sum_{n=0}^{\infty} (2 + a_n)^n z^n$.
2. (30p) Let $\gamma$ be the closed curve given by the ellipse $x^2 + \frac{y^2}{4} = 1$ traversed once in the positive (counterclockwise) direction. Compute

$$\int_{\gamma} \frac{e^{iz^3}}{z^2 + 1} \, dz.$$
3. (30p) Let $\gamma$ be the half-circle $\gamma(t) = e^{it}$, $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Compute
\[ \int_{\gamma} ze^{iz} \, dz. \]
4. (30pt) For $a \in (-1, 1)$, let $D_a = \{z : |z| < 1, \ \text{Re} \ z > a\}$. For each such $a$, either find a Möbius transformation of $D_a$ onto the quadrant $Q = \{w = re^{i\theta} : r > 0, \ 0 < \theta < \frac{\pi}{2}\}$, or show that such cannot exist.
5. (15+15=30p) Let $f$ be analytic in the open disk $B(0, 2)$ and continuous in the closed disk $\overline{B}(0, 2)$.

(a) Show that $\sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}$ converges uniformly in the closed disk $\overline{B}(0, c)$ for every $c < 1$.

(b) Show that the function defined by this series is analytic in the unit disk $B(0, 1)$.

Note: A result from a homework problem is relevant to (b). If you use it, you must reprove the result.
6. (30p) Let $f$ be analytic in the open disk $B(0, 1+\delta)$ with $\delta > 0$. Assume that $|f(z)| = 1$ on the unit circle $|z| = 1$ and that $f$ has $m$ zeros, counted with multiplicities, in the unit disk $\mathbb{D} = B(0, 1)$. Show that the equation $f(z) = \alpha$ has exactly $m$ roots, counted with multiplicities, for every $\alpha \in \mathbb{D}$. 