Solutions: Homework 1

Nandagopal Ramachandran

October 11, 2019

**Problem 1.** If \((X, d)\) is any metric space show that every open ball is, in fact, an open set. Also, show that every closed ball is a closed set.

**Proof.** Let \(U = B(x; r)\) denote the open ball in \(X\) with center \(x\) and radius \(r\). Let \(y \in U\) and \(d(x, y) = r - \epsilon\) for some \(0 < \epsilon \leq r\). Then an application of the triangle inequality shows that \(B(y; \epsilon) \in U\). So \(U\) is open. Now, let \(V = \overline{B}(x; r)\) denote the closed ball in \(X\) with center \(x\) and radius \(r\). We will show that \(V^c\) is open. Let \(y \in V^c\) and \(d(x, y) = r + \epsilon\) for some \(\epsilon > 0\). Then \(B(y; \epsilon) \in V^c\) by the triangle inequality. So \(V^c\) is open, hence \(V\) is closed.

**Problem 2.** Let \((X, d)\) be a metric space. Then:
(a) The sets \(X\) and \(\varnothing\) are closed;
(b) If \(F_1, \ldots, F_n\) are closed sets in \(X\) then so is \(\bigcup_{k=1}^n F_k\);
(c) If \(\{F_j : j \in J\}\) is any collection of closed sets in \(X\), \(J\) any indexing set, then \(F = \cap\{F_j : j \in J\}\) is also closed.

**Proof.** (a) \(X^c = \varnothing\) and \(\varnothing^c = X\) and since \(\varnothing\) and \(X\) are open, they should also be closed.

(b) By definition, \(F_1^c, \ldots, F_n^c\) are open sets in \(X\). Since finite intersections of open sets are open, \(\cap_{k=1}^n F_k^c\) is open. So \(\bigcup_{k=1}^n F_k = (\cap_{k=1}^n F_k)^c\) is closed.

(c) Again, by definition, we have \(\{F_j^c : j \in J\}\) is an arbitrary collection of open sets in \(X\). Since arbitrary unions of open sets are open, \(F^c = \cup\{F_j^c : j \in J\}\) is open. So \(F\) is closed.

**Problem 3.** Show that \((\mathbb{C}_\infty, d)\) where \(d\) is given by

\[
d(z, z') = \frac{2|z - z'|}{[(1 + |z|^2)(1 + |z'|^2)]^{\frac{1}{2}}} \quad (z, z' \in \mathbb{C})
\]

and

\[
d(z, \infty) = \frac{2}{(1 + |z|^2)^{\frac{1}{2}}}
\]

is a metric space.
Proof. Following the notation in §6 of Chapter 1, let $f : \mathbb{C}_\infty \to S$ denote the inverse of the stereographic projection. Let $\text{dist}$ denote the Euclidean metric on $\mathbb{R}^3$, restricted to $S$ in this case. Then we know that $d(z, z') = \text{dist}(f(z), f(z'))$, by construction. Since $\text{dist}$ is a metric, $d(z, z') = \text{dist}(f(z), f(z')) \geq 0$. Suppose $d(z, z') = 0$. Then $\text{dist}(f(z), f(z')) = 0$, hence $f(z) = f(z')$, and so $z = z'$ as $f$ is bijective. Again, the symmetry and triangle inequality just carries over from $\text{dist}$ to $d$. So $d$ is a metric. 

Problem 4. Let $A$ and $B$ be subsets of a metric space $(X, d)$. Then:

(a) $A$ is open if and only if $\overline{A} = \text{int} A$;
(b) $A$ is closed if and only if $A = \overline{A}$;
(c) $\text{int} A = X - (X - A); \overline{A} = X - \text{int}(X - A); \partial A = \overline{A} - \text{int} A$;
(d) $(A \cup B) = \overline{A} \cup \overline{B}$;
(e) $x_0 \in \text{int} A$ if and only if there is an $\epsilon > 0$ such that $B(x_0; \epsilon) \subset A$.

Proof. (a) Since $\text{int} A$ is open, if $\text{int} A = A$, then $A$ is open. Conversely, if $A$ is open, then $A$ is an element of $\{G : G$ is open and $G \subset A\}$. So $\text{int} A = A$.

(b) Since $\overline{A}$ is closed, if $\overline{A} = A$, then $A$ is closed. Conversely, if $A$ is closed, then $A$ is an element of $\{F : F$ is closed and $F \supset A\}$. So $\overline{A} = A$.

(c) Since $\text{int} A \subset A$, we have $X - A \subset X - \text{int} A$. Since $X - \text{int} A$ is closed, we have $(X - A) \subset X - \text{int} A$. Similarly, since $X - A \subset (X - A)$, so $X - (X - A) \subset A$. Since $X - (X - A)$ is open, we have $X - (X - A) \subset A$. So $X - \text{int} A \subset (X - A)$. This gives us the first equality. Now if we replace $A$ by $X - A$ in the first one, we get the second one. Now $\overline{A} - \text{int} A = \overline{A} \cap (X - \text{int} A) = \overline{A} \cap (X - A)$ (by the first equality) = $\partial A$.

(d) Since $A \subset \overline{A}$ and $B \subset \overline{B}$, we have $A \cup B \subset \overline{A} \cup \overline{B}$. Since $\overline{A} \cup \overline{B}$ is closed, we have $\overline{A} \cup \overline{B} \subset \overline{A} \cup \overline{B}$. Now $A \subset A \cup B$, so $\overline{A} \subset (A \cup B)$. Similarly, $\overline{B} \subset (A \cup B)$. So $\overline{A} \cup \overline{B} \subset (A \cup B)$. This gives equality.

(e) If $x_0 \in \text{int} A, \exists \epsilon > 0$ such that $B(x_0; \epsilon) \subset \text{int} A$, as $\text{int} A$ is open. Since $\text{int} A \subset A$, we have $B(x_0; \epsilon) \subset A$. On the other hand, if $\exists \epsilon > 0$ such that $B(x_0; \epsilon) \subset A$, then since $B(x_0; \epsilon)$ is open, $B(x_0; \epsilon) \subset \text{int} A$, hence $x_0 \in A$.

Problem 5. The purpose of this exercise is to show that a connected subset of $\mathbb{R}$ is an interval.

(a) Show that a set $A \subset \mathbb{R}$ is an interval iff for any two points $a$ and $b$ in $A$ with $a < b$, the interval $[a, b] \subset A$.

(b) Use part (a) to show that if a set $A \subset \mathbb{R}$ is connected then it is an interval.

Proof. (a) It is clear that if $A$ is an interval, then the given criterion holds. Conversely, suppose this holds. Let $c = \inf A$ and $d = \sup A$. Note that these could be $-\infty$ and $+\infty$ respectively. If $c = d$, it is clear that $A = \{c\}$, which is an interval. So, suppose $c < d$. We claim that the interval $(c, d) \subset A$. Suppose not. Then $\exists x \in (c, d)$ such that $x \not\in A$. Since $x$
is neither the supremum, nor the infimum of $A$, $\exists s < x < t$ with $s, t \in A$. But by the given criterion, $[s, t] \subset A$. In particular, $x \in A$, a contradiction. So $(c, d) \subset A$. Now depending on whether $c$ and/or $d$ are contained in $A$, we see that $A$ is either an open, semi-open or closed interval.

(b) Suppose $A$ is connected, but not an interval. Then, by part (a), $\exists a, b \in A$ such that $\exists x \in [a, b]$, but $x \notin A$. Then the set $A \cap (-\infty, x] = A \cap (-\infty, x)$ is a proper, non-empty subset of $A$ that is both open and closed. This contradicts the fact that $A$ is connected. So $A$ is an interval.

**Problem 6.** Prove that if $\{D_j : j \in J\}$ is a collection of connected subsets of $X$ and if for each $j$ and $k$ in $J$ we have $D_j \cap D_k \neq \emptyset$ then $D = \cup \{D_j : j \in J\}$ is connected.

**Proof.** Suppose that $D$ is not connected. Choose a non-empty proper subset $A$ of $D$ that is both open and closed. Since the $D_j$'s are connected and $A \cap D_j$ is both open and closed in $D_j$, we get that $A \cap D_j = \emptyset$ or $D_j$. Choose $j$ and $k$ in $J$ such that $A \cap D_j = D_j$ and $A \cap D_k = \emptyset$. Note that we can find such a pair $j$ and $k$ because otherwise, we either have $A \cap D_j = \emptyset \forall j \in J$ or $D_j \subset D - A \forall j \in J$, which would imply that $A = D$ or $A = \emptyset$ respectively. Then $(D - A) \cap D_j = \emptyset$ and $(D - A) \cap D_k = D_k$. Then $D_j \cap D_k = (A \cap D_k \cap D_j) \cup ((D - A) \cap D_j \cap D_k) = (\emptyset \cap D_j) \cup (\emptyset \cap D_k) = \emptyset$, which contradicts our assumption. \qed