Solutions: Homework 3

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**Problem 1.** We say that \( f : X \to \mathbb{C} \) is bounded if there is a constant \( M > 0 \) with \(|f(x)| \leq M \) for all \( x \) in \( X \). Show that if \( f \) and \( g \) are bounded uniformly continuous (Lipschitz) functions from \( X \) into \( \mathbb{C} \) then so is \( fg \).

**Proof.** Let \( d \) denote the metric on \( X \). Since \( f \) and \( g \) are bounded, there exists \( M > 0 \) such that \(|f(x)| \leq M \) and \(|g(x)| \leq M \) for all \( x \) in \( X \). So, \(|(fg)(x)| \leq M^2 \) for all \( x \) in \( X \) and hence \( fg \) is bounded. Now, let \( \epsilon > 0 \). By the uniform continuity of \( f \) and \( g \), there exists \( \delta > 0 \) such that \(|f(x) - f(y)| < \epsilon/2M \) and \(|g(x) - g(y)| < \epsilon/2M \) for all \( x, y \) in \( X \) such that \( d(x, y) < \delta \).

Then, for any \( x, y \) in \( X \) such that \( d(x, y) < \delta \), we have

\[
|f(x)g(x) - f(y)g(y)| = |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|
\]

\[
\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \leq M \frac{\epsilon}{2M} + M \frac{\epsilon}{2M} = \epsilon
\]

This proves the uniform continuity of \( fg \).

Now, let \( f \) and \( g \) be bounded (with bound \( M \)) Lipschitz functions with constant \( M' \). Then \(|f(x) - f(y)| \leq M'd(x, y) \) and \(|g(x) - g(y)| \leq M'd(x, y) \) for all \( x, y \) in \( X \). Then, as above,

\[
|f(x)g(x) - f(y)g(y)| \leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)|
\]

\[
\leq MM'd(x, y) + MM'd(x, y) = 2MM'd(x, y)
\]

So, \( fg \) is Lipschitz with constant \( 2MM' \).

\[ \square \]

**Problem 2.** Suppose \( f : X \to \Omega \) is uniformly continuous; show that if \( \{x_n\} \) is a Cauchy sequence in \( X \) then \( \{f(x_n)\} \) is a Cauchy sequence in \( \Omega \). Is this still true if we only assume that \( f \) is continuous?

**Proof.** Let \( d \) denote the metric on \( X \) and let \( \rho \) denote the metric on \( \Omega \). Let \( \epsilon > 0 \). Then, by the uniform continuity of \( f \), there exists \( \delta > 0 \) such that \( \rho(f(x), f(y)) < \epsilon \) whenever \( d(x, y) < \delta \). By the Cauchy-ness of \( \{x_n\} \), there exists \( N \in \mathbb{N} \) such that \( d(x_n, x_m) < \delta \) for all \( n, m \geq N \). This implies that \( \rho(f(x_n), f(x_m)) < \epsilon \) for all \( n, m \geq N \). As \( \epsilon > 0 \) was arbitrary, we conclude that \( \{f(x_n)\} \) is Cauchy in \( \Omega \).

This is not true if \( f \) is just assumed to be continuous. For example, take \( f : (0, 1) \to (1, \infty) \) given by \( f(x) = 1/x \). Then the sequence \( \{1/n\} \) is Cauchy in \( (0, 1) \) but \( \{f(1/n)\} \) is not Cauchy in \( (1, \infty) \).

\[ \square \]
Problem 3. Suppose that $\Omega$ is a complete metric space and that $f : (D, d) \to (\Omega, \rho)$ is uniformly continuous, where $D$ is dense in $(X, d)$. Use Problem 2 to show that there is a uniformly continuous function $g : X \to \Omega$ with $g(x) = f(x)$ for every $x$ in $D$.

Proof. Let $x$ in $X$. We can then choose a sequence $\{x_n\}$ in $D$ that converges to $x$ in $X$. Since $\{x_n\}$ is a Cauchy sequence (because it is convergent), by Problem 2, we know that $\{f(x_n)\}$ is a Cauchy sequence in $\Omega$. Since $\Omega$ is complete, it converges in $\Omega$ to a limit, which we shall denote by $g(x)$. Now, let $\{y_n\}$ be another sequence in $D$ converging to $x$ in $X$. Then it is easy to see that the sequence $x_1, y_1, x_2, y_2, \ldots$ is a Cauchy sequence in $D$ converging to $x$ in $X$. So, the sequence $f(x_1), f(y_1), f(x_2), f(y_2), \ldots$ in $\Omega$ is Cauchy and has a convergent subsequence $\{f(x_n)\}$ converging to $g(x)$. This implies that the subsequence $\{f(y_n)\}$ also converges to $g(x)$. So, $g(x)$ is an element of $\Omega$ that is dependent only on $x$ and not on the choice of sequence in $D$. So the function $g : X \to \Omega$ is well-defined. Clearly, if $x \in D$, choosing the sequence $\{x_n = x\}$ in $D$ implies that $g(x) = f(x)$. Now, let $\epsilon > 0$. Since $f$ is uniformly continuous, there exists $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ whenever $x, y$ in $D$ with $d(x, y) < \delta$. Let $x, y$ in $X$ be such that $d(x, y) = \delta - r$ with $0 < r \leq \delta$. Choose 2 sequences $\{x_n\}$ and $\{y_n\}$ in $D$ converging to $x$ and $y$, respectively. Choose $N$ large enough such that $d(x_N, x) < r/2, d(y_N, y) < r/2, \rho(f(x_N), g(x)) < \epsilon$ and $\rho(f(y_N), g(y)) < \epsilon$. Then

$$d(x_N, y_N) \leq d(x_N, x) + d(x, y) + d(y, y_N) < r/2 + \delta - r + r/2 = \delta.$$ 

This implies that $\rho(f(x_N, y_N)) < \epsilon/3$. So,

$$\rho(g(x), g(y)) \leq \rho(g(x), f(x_N)) + \rho(f(x_N), f(y_N)) + \rho(f(y_N), g(y)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$ 

This shows that $g$ is uniformly continuous.

Problem 4. Let $\{f_n\}$ be a sequence of uniformly continuous functions from $(X, d)$ into $(\Omega, p)$ and suppose that $f = u\lim f_n$ exists. Prove that $f$ is uniformly continuous. If each $f_n$ is a Lipschitz function with constant $M_n$ and $\sup M_n < \infty$, show that $f$ is a Lipschitz function. If $\sup M_n = \infty$, show that $f$ may fail to be Lipschitz.

Proof. Let $\epsilon > 0$. Fix $N$ large enough such that $p(f_N(x), f(x)) < \epsilon/3$ for all $x$ in $X$. Since $f_N$ is uniformly continuous, there exists $\delta > 0$ such that $p(f_N(x), f_N(y)) < \epsilon/3$ for all $x, y$ in $X$ with $d(x, y) < \delta$. Then, for all $x, y$ in $X$ with $d(x, y) < \delta$, we have

$$p(f(x), f(y)) \leq p(f(x), f_N(x)) + p(f_N(x), f_N(y)) + p(f_N(y), f(y)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$ 

So $f$ is uniformly continuous.

Now suppose the $f_n$’s are Lipschitz functions with constant $M_n$. So $p(f_n(x), f_n(y)) \leq M_n d(x, y)$ for all $x, y$ in $X$. Let $M = \sup M_n$. Pick $N$ large enough such that $p(f(x), f_N(x)) < \epsilon/2$ for all $x$ in $X$. Then, we have

$$p(f(x), f(y)) \leq p(f(x), f_N(x)) + p(f_N(x), f_N(y)) + p(f_N(y), f(y)) < \epsilon/2 + M_N d(x, y) + \epsilon/2 \leq \epsilon + M d(x, y).$$
So we have that \( p(f(x), f(y)) - Md(x, y) < \epsilon \) for all \( x, y \) in \( X \). As \( \epsilon > 0 \) was arbitrary, we have \( p(f(x), f(y)) \leq Md(x, y) \). Thus, \( f \) is a Lipschitz function.

Now, we use the fact that every 2\( \pi \)-periodic continuous function on \([-\pi, \pi]\) can be approximated uniformly by trigonometric polynomials, i.e. \( f: [-\pi, \pi] \to \mathbb{C} \) is a continuous function with \( f(-\pi) = f(\pi) \), then there exists a sequence \( \{f_n\} \) of trigonometric polynomials that converge uniformly to \( f \). If we prove that every trigonometric polynomial is Lipschitz, then taking any continuous 2\( \pi \)-periodic function \( f: [-\pi, \pi] \to \mathbb{C} \) that is not Lipschitz gives us a counterexample. For \( k \in \mathbb{Z} \), let \( g_k : [-\pi, \pi] \to \mathbb{C} \) be given by \( g_k(x) = e^{ikx} \). Then, for \( x \neq y \),

\[
\frac{|g_k(x) - g_k(y)|}{|x - y|} = \frac{|e^{ikx} - e^{iky}|}{|x - y|} = 2\left|\sin \frac{x}{2}(x - y)\right| \leq k
\]

So \( g_k \) is Lipschitz with constant \( k \). Since a finite linear combination of Lipschitz functions is Lipschitz, any trigonometric function is Lipschitz. As an example, we take \( f(x) = |x| \ln(|x|) \). If \( f \) is Lipschitz, there exists \( M > 0 \) such that \( \frac{|f(x) - f(0)|}{|x - 0|} < M \) for all \( x \in [-\pi, \pi], x \neq 0 \). But \( \frac{|f(x) - f(0)|}{|x - 0|} = |\ln |x|| \) which is unbounded near 0. So \( f \) is not Lipschitz. Note that by observing that polynomials on a bounded interval are Lipschitz, we could also apply Weierstrass approximation theorem to obtain counterexamples.

\[ \square \]

**Problem 5.** Show that the radius of convergence of the power series

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}
\]

is 1, and discuss convergence for \( z = 1, -1, \) and \( i \).

**Proof.** For this power series, \( a_n = \frac{(-1)^m}{m} \) if \( n = m(m + 1) \) for some \( m \in \mathbb{N} \) and 0 otherwise.

\[
\limsup |a_n|^{1/n} = \limsup \left| \frac{(-1)^n}{n} \right|^{1/(n+1)} = \limsup \frac{1}{n^{1/(n+1)}} = \lim \frac{1}{n^{1/(n+1)}}
\]

\[
= \frac{1}{\lim n^{1/(n+1)}} = \frac{1}{\lim \frac{\ln n}{n(1+1)}} = \frac{1}{e^0} = 1.
\]

So \( 1/R = 1 \), hence \( R = 1 \).

Since \( n(n+1) \) is even for all \( n \geq 1 \), for \( z = 1, -1 \), the series equals \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \ln 2 \). Let \( z = i \). Then the series becomes

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} i^{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (-1)^{n(n+1)/2} = \sum_{n=1}^{\infty} \frac{(-1)^{n(n+3)/2}}{n} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n(2n-1)}
\]

By the alternating series test, this converges. \[ \square \]