Problem 1. Let $G = \mathbb{C} \setminus \{0\}$ and show that every closed curve in $G$ is homotopic to a closed curve whose trace is contained in $\{z : |z| = 1\}$.

Proof. Let $\gamma : [0, 1] \to G$ be a closed curve in $G$. Let $\gamma' : [0, 1] \to \{z : |z| = 1\}$ be the curve given by

$$\gamma'(s) = \frac{\gamma(s)}{|\gamma(s)|}.$$

This is well-defined because $\gamma(s) \neq 0$ for all $s \in [0, 1]$. We will show that $\gamma$ and $\gamma'$ are homotopic. Define $\Gamma : [0, 1] \times [0, 1] \to G$ by

$$\Gamma(s, t) = (1 - t)\gamma(s) + t\gamma'(s)$$

Note that $\Gamma(s, t) \neq 0$ for all $s, t$ and so it is well-defined. It is clearly continuous and

$$\Gamma(s, 0) = \gamma(s), \quad \Gamma(s, 1) = \gamma'(s)$$

$$\Gamma(0, t) = \Gamma(1, t) \quad (0 \leq t \leq 1).$$

So $\gamma$ is homotopic to $\gamma'$ in $G$.

Problem 2. Let $G = \mathbb{C} \setminus \{a, b\}, a \neq b$, and let $\gamma$ be the curve in the book. Show that $n(\gamma; a) = n(\gamma; b) = 0$.

Proof. This solution is not rigorous. We see that there are two closed curves going around $a$ with one going in the clockwise direction and the other in the anti-clockwise direction. This means that the index contributed by one of them is 1 and the other one is -1. Adding up, we see that $n(\gamma; a) = 0$. The same argument holds for $b$.

Problem 3. Let $G$ be a region and let $\gamma_0$ and $\gamma_1$ be two closed smooth curves in $G$. Suppose $\gamma_0 \sim \gamma_1$ and $\Gamma$ satisfies (6.2). Also suppose that $\gamma_t(s) = \Gamma(s, t)$ is smooth for each $t$. If $w \in \mathbb{C} \setminus G$ define $h(t) = n(\gamma_t; w)$ and show that $h : [0, 1] \to \mathbb{Z}$ is continuous.

Proof. Since $[0, 1]$ is connected, this is equivalent to showing that $h$ is constant. We know that, by Cauchy’s theorem, if $\gamma$ and $\gamma'$ are two homotopic closed rectifiable curves in $G$, then $n(\gamma; w) = n(\gamma'; w)$ for all $w \in \mathbb{C} \setminus G$. We will prove that for all $t \in [0, 1], \gamma_0$ is homotopic to $\gamma_t$. This shows that $h(t) = 0$ for all $t \in [0, 1]$, and hence $h$ is constant. Fix $0 \leq t_0 \leq 1$. Let $\Gamma' : [0, 1] \times [0, 1] \to G$ by

$$\Gamma'(s, t) = \Gamma(s, t_0 t)$$

Then $\Gamma'$ is a homotopy from $\gamma_0$ to $\gamma_{t_0}$. This concludes our proof.
Problem 4. Let \( G \) be open and suppose that \( \gamma \) is a closed rectifiable curve in \( G \) such that \( \gamma \approx 0 \). Set \( r = d(\{ \gamma \}, \partial G) \) and \( H = \{ z \in \mathbb{C} : n(\gamma; z) = 0 \} \).

(a) Show that \( \{ z : d(z, \partial G) < \frac{1}{2}r \} \subset H \).

(b) Use part (a) to show that if \( f : G \to \mathbb{C} \) is analytic then \( f(z) = \alpha \) has at most a finite number of solutions \( z \) such that \( n(\gamma; z) \neq 0 \).

Proof. (a) Let \( z \) be such that \( d(z, \partial G) < \frac{1}{2}r \). Then there exists \( x \in \partial G \) such that \( d(z, x) < \frac{1}{2}r \). Then \( B(x; \frac{1}{2}r) \) is a connected subset of \( \mathbb{C} \setminus \{ \gamma \} \). Then \( n(\gamma; \cdot) \) is constant on \( B(x; \frac{1}{2}r) \). But \( B(x; \frac{1}{2}r) \cap (\mathbb{C} \setminus G) \neq \emptyset \). Since \( \gamma \approx 0 \), this shows that \( n(\gamma; z) = 0 \). As \( z \) is arbitrary, this completes the proof.

(b) WLOG, assume that \( \alpha = 0 \). Assume that \( f \) is not the constant function. Let \( Z = \{ z \in G : f(z) = 0 \} \). Then \( Z \) has no limit points in \( G \), by Theorem 3.7. This implies that any limit point lies in \( \partial G \). Now we know that the set \( \{ z \in \mathbb{C} : n(\gamma; z) \neq 0 \} \) is bounded. Suppose there exists infinitely many \( z \in G \) such that \( f(z) = 0 \) and \( n(\gamma; z) \neq 0 \), and denote the set of all such \( z \) by \( V \). The set \( \{ z \in \mathbb{C} : n(\gamma; z) \neq 0 \} \) is bounded. So \( V \) is bounded, hence \( \overline{V} \) is compact. So there exists a sequence \( \{ x_n \} \) in \( V \) that converges to \( x \) in \( \overline{V} \). But we know that \( Z \) and hence \( V \) has no limit points in \( G \). So \( x \in \partial G \). Then, by continuity, \( n(\gamma; x) \neq 0 \), which contradicts (a).

Problem 5. Let \( f \) be analytic in \( B(a; R) \) and suppose that \( f(a) = 0 \). Show that \( a \) is a zero of multiplicity \( m \) iff \( f^{(m-1)}(a) = \ldots = f(a) = 0 \) and \( f^{(m)}(a) \neq 0 \).

Proof. Suppose that \( a \) is a zero of multiplicity \( m \). Then there exists an analytic function \( g : B(a; R) \to \mathbb{C} \) such that \( f(z) = (z - a)^m g(z) \) where \( g(a) \neq 0 \). Then, \( h(z) = (z - a)^{m-1} g(z) \) has a zero of multiplicity \( m - 1 \) at \( a \). Inductively, we assume that \( h^{(m-2)}(a) = \ldots = h(a) = 0 \) and \( h^{(m-1)}(a) \neq 0 \). \( f(z) = (z - a) h(z) \). So \( f^{(i)}(z) = (z - a) h^{(i)}(z) + \sum_{j=0}^{i-1} h^{(j)}(z) \). Then we see that \( f^{(m-1)}(a) = \ldots = f(a) = 0 \) and \( f^{(m)}(a) \neq 0 \).

Conversely, suppose \( f^{(m-1)}(a) = \ldots = f(a) = 0 \) and \( f^{(m)}(a) \neq 0 \). Let \( a \) be a zero of multiplicity \( k \). Then \( f^{(k)}(a) \neq 0 \), hence \( k \geq m \), but \( f^{(i)}(a) = 0 \) for \( i < k \) by the above paragraph. This implies that \( k \leq m \). So \( k = m \).

Problem 6. Suppose that \( f : G \to \mathbb{C} \) is analytic and one-one; show that \( f'(z) \neq 0 \) for any \( z \) in \( G \).

Proof. Suppose \( f'(a) = 0 \) for some \( a \in G \). Let \( g : G \to \mathbb{C} \) be defined as \( g(z) = f(z) - f(a) \). Then \( g(a) = g'(a) = 0 \). So \( g \) has a zero at \( a \) of multiplicity at least 2, say, \( m \). Then, by Theorem 7.4, there exists \( \epsilon > 0 \) and \( \delta > 0 \) such that for \( 0 < |\zeta| < \delta \), the equation \( g(z) = \zeta \) has exactly \( m \) simple roots in \( B(a; \epsilon) \). This contradicts the fact that \( g \) is one-one.