Cauchy's Thm and Integral Formula

Cauchy's Integral Formula - I. Let \( f \) be analytic in \( G \subseteq \mathbb{C} \) and \( \gamma: [0,1] \rightarrow G \) a p.-w. smoothly closed curve s.t. \( n(\gamma; 0) = 0, \forall z \in C \smallsetminus G \). Then, \( \forall z \in G \setminus \gamma \) holds:

\[
f(z) n(\gamma, z) = \pm \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z}.
\]

\( D_0 = \gamma(0) \smallsetminus \gamma \) open!

\( \text{Pf.} \) Note, by assumption, \( G = G \cup D_0, D_0 = \{z \in C \smallsetminus G; n(\gamma; z) = 0\} \). We define on \( G \):

\[
g(z) = \begin{cases}
\pm \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta, & z \in G \\
\pm \frac{1}{2\pi i} \int_{\gamma_z} f(\zeta) d\zeta, & z \in D_0
\end{cases}
\]

(see below for def. \( z \in \gamma \))

Claim. \( g(z) \) entire (analytic in \( G \)).

- First, check \( g \) is well-defined. Pth \( z \in G \cap D_0 \):

\[
\pm \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta = \pm \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z} - f(z) \mp \frac{1}{2\pi i} \int_{\gamma_z} \frac{f(\zeta) d\zeta}{\zeta - z} = \pm \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) d\zeta}{\zeta - z}.
\]

\( n(\gamma; 0) = 0 \) in \( D_0 \)

\( g \) is analytic in \( D_0 \) by C-Leibniz.

To see \( g \) analytic in \( G \), we consider \( \varphi(z, 3) := \frac{f(\zeta) - f(3)}{\zeta - z} \) in \( G \times G \),

where \( \varphi(z, 3) = \frac{f(\zeta)}{\zeta - z} \). Then, \( \varphi \) is cont. in \( G \times G \). We claim \( \forall z \in G, \varphi(z) = \varphi(z, 3) \) is analytic. \( \varphi \) is clearly anal. in \( G \smallsetminus \{3\} \). Suffices to check \( \varphi \) is anal. in \( B(3, r) \), where \( B(3, r) \subseteq G \). This is easy:

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - 3)^n; \quad a_n = \frac{f^{(n)}(3)}{n!}; \quad f(z) - f(3) = \sum_{n=1}^{\infty} a_n (z - 3)^n
\]

\( \Rightarrow \sum_{n=1}^{\infty} a_n (z - 3)^n = (z - 2) h(z), \) where \( h \) is anal. in \( B(3, 1) \).
\[ h(z) = \sum_{n=0}^{\infty} c_n (z-3)^n = (z-3) h(z), \] where \( h \) is anal. in \( D^{\alpha \beta} \).

\[ h(z): R, 0 < R < 1 \]

But \( h(z) = \frac{\phi(z) - \phi(z)}{z-3} = h(z) \Rightarrow \phi \) anal. in \( G \). The *precise* def of \( g \):

\[ g(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\phi(w)}{w-3} dw \text{ is analytic in } G \] by C.Leradziev.

Claim 2. \( g \) is constant and, in fact, \( g \equiv 0 \).

The unbd comp. \( D_0 \) of \( G \setminus \{ \infty \} \) is contained in \( D_0 \). As in

that proof, when \( |z| \to \infty \) (\( z \in D_0 \) eventually),

\[ |g(z)| \leq \frac{1}{2\pi} \int_{\partial D} \left| \frac{\phi(w)}{w-3} \right| dw \leq \left( \max_{|w| \leq \partial D} \frac{1}{3} \frac{1}{|w-3|} \right) M \cdot \text{length of } \partial D \]

\[ \leq \max_{|w| \leq \partial D} |\phi(w)| \int_{|w| = \infty} |dw| \]

0 as \( |z| \to \infty \).

In particular, \( |g(z)| \leq 1 \text{ in } G \setminus \{ \infty \}; \) \( |g(z)| = \max \{ 1, \max_{|w| \leq \partial D} |\phi(w)| \} \)

\( \Rightarrow g \) is constant by Liouville. But \( g(z) \to 0 \text{ as } z \to \infty \Rightarrow g \equiv 0, \)

as claimed.

Thus, if \( z \in G \setminus \{ \infty \} \), we get

\[ 0 = \frac{1}{2\pi i} \int_{\partial D} \frac{\phi(z) - \phi(z)}{z-3} dw = \frac{1}{2\pi i} \int_{\partial D} \frac{\phi(z)}{z-3} dw - \frac{1}{2\pi i} \int_{\partial D} \frac{\phi(z)}{3-z} \frac{1}{n(z)} \]

**CIF-2.** \( \phi \) anal. in \( G \), \( \chi_1, \ldots, \chi_n \) closed, p-w smooth curves in \( G \)

s.t. \( \nu(\chi_1, z) \ldots \nu(\chi_n, z) = 0 \) in \( G \). Then, \( \forall z \in G \).

\[ \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\partial \chi_k} \frac{\phi(z)}{3-z} \]

\[ \left( \sum_{k=1}^{n} \nu(\chi_k, z) \right) \phi(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\partial \chi_k} \frac{\phi(z)}{3-z} . \]
Pf. Just use $y = y_1 + \cdots + y_n$ in Pf above.\[\square\]

Cauchy's Thm - I. Let $f, g, y_1, \ldots, y_n$ be as in C1F-\text{-II.}$

Then
$$\sum_{k=1}^{n} \oint_{\partial \mathcal{D}_k} f \, dz = 0$$

Pf. Pick $a \in G \setminus \mathcal{D}_n$ and consider $g(z) = (z-a) f(z)$ in C1F-\text{-II.} \[\square\]

Typical Ex. Consider 2 closed curves $\partial G$, be shaded region \text{(w/ hole inside $\partial \mathcal{D}_1$). Then, $n_{-\partial \mathcal{D}_1} + n_{\partial \mathcal{D}_2} = 0$ in $G \setminus \mathcal{D}_1 \cup \mathcal{D}_2$. Thus, if $f$ anal. in $G$,

$$\left( n(\partial_1, z) - n(\partial_2, z) \right) f(z) = \frac{1}{2\pi i} \oint_{\partial_1} \frac{f(z) \, dz}{z-z} - \frac{1}{2\pi i} \oint_{\partial_2} \frac{f(z) \, dz}{z-z}$$

Converse to CT-I.

Morera's Thm. Let $f$ be cont. in $G$. If, for every $B(a,r) \subseteq G$
and every triangular path $\Gamma = \left[ a_1, a_2 \right] \cup \left[ a_2, a_3 \right] \cup \left[ a_3, a_1 \right]$ in $B(a,r)$, we have $\oint_{\Gamma} f(z) \, dz = 0$ , then $f$ is analytic in $G$.

Pf. Suffices to show $f$ analyt. in $B(a,r) \subseteq G$. We shall show
3 anal. for $F$ in $B(a,r)$ s.t. $F = f$. This completes proof.

Set $F(z) = \oint_{\partial \mathcal{D}} f(\mathcal{D})$
By assumption, \( \int_{a,2}^b d3 = \int_{a,2}^{b,2} d3 + \int_{b,2}^{2,2} d3 \Rightarrow \)

\[
\frac{F(z) - F(z_0) - f(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{z_0,2}^{z,2} (f(z) - f(z_0)) d3 .
\]

\[ \forall \exists 0 \exists \delta > 0 \text{ s.t. } \max_{z \in [z_0,2]} |f(z) - f(z_0)| < \epsilon \text{ when } |z - z_0| < \delta \Rightarrow \]

\[ |\frac{F(z) - F(z_0) - f(z_0)}{z - z_0}| < \epsilon \text{ when } |z - z_0| < \delta \Rightarrow F \text{ C-diff at } z_0 \text{ and } F'(z_0) = f(z_0) . \text{ Since } F \text{ is cont, } z_0 \in B(a, r) \text{ arbitrary, } \Rightarrow F \text{ analyh2 in } B(a, r) \Rightarrow F = F' \text{ analyh2 in } B(a, r) . \]