Recall. • **Argument principle.** Assume \( f \) is 0 and \( f \) anal. in \( G \). If \( f(z) = 0 \) has roots \( a_1, \ldots, a_n \) in \( G \), then

\[
\frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z)-0} \, dz = \sum_{k=1}^{n} \frac{1}{\rho(a_k)}.
\]

Typical application: \( f \) is simple w/ \( G \backslash \{a\} = \{v(z) = 0 \cup u(z) = 1 \} \).

Then, \( \frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)}{f(z)-0} \, dz \neq \emptyset \{ \text{roots w/ multi in } G \} \).

• **Local behavior of** \( w = f(z) \). Suppose \( f(z) = 0 \) has root of mult. \( m \equiv 1 \) at \( a \in G \). Then, \( \exists \delta > 0 \) s.t. \( f(z) = 0 \) has \( m \) simple (mult = 1) roots in \( \mathbb{B}(a, \delta) \) for each \( a \in \mathbb{B}(a, \epsilon) \).

In particular, \( \mathbb{B}(a, \epsilon) \subseteq f(\mathbb{B}(a, \delta)) \). Thus, if \( G \) is a region, \( f \) nonconstant

\( \Rightarrow \) for every \( a \in \mathbb{B}(a) \) and \( a \notin f^{-1}(\{0\}) \), \( f(z) = 0 \) has root of finite mult at \( z = a \).  

\( \Rightarrow \exists \mathbb{B}(a, \epsilon) \subseteq f(G) \Rightarrow f(G) \) open.

• Do Open Mapping Theorem + Cor 1 from Lecture 24 notes.

**Goursat’s Thm.** Let \( G \subseteq \mathbb{C} \) and assume \( f \) is \( C \)-diff. at every \( a \in G \).

Then \( f \) is anal. in \( G \).

**Pf.** Use Morera’s Thm. Pick \( B(a, \epsilon) \subseteq G \), and triangular path \( T \subseteq B(a, \epsilon) \).

\( \Rightarrow \exists \Delta \), \( \Delta \) closed triangle.
We have \( \sum_{k=1}^{4} \frac{1}{T_k} \int \frac{f}{T_k} \, dz \) by cancellation over interior segments. Pick \( T^{(n)} \) to be \( T_k \) s.t. \( \frac{1}{4} \int \frac{f}{T_k} \, dz \leq \max_k \frac{1}{T_k} \int \frac{f}{T_k} \, dz \).

\[ \frac{|Sf|}{T} \leq 4 \frac{|Sf|}{T^{(n)}}. \quad \text{Note: } l(T^{(n)}) = \frac{1}{2} l(T), \quad \text{diam}(\Delta^{(n)}) = \frac{1}{2} \text{diam}(\Delta). \]

Repeate. Inductively, we obtain \( \Delta^{(0)} = \Delta \supseteq \Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \ldots \). closed triangles. We have \( l(T^{(n)}) = 2^{-n} l(T) \), \( \text{diam}(\Delta^{(n)}) = 2^{-n} \text{diam}(\Delta). \) By Cantor's Paradox, \( \bigcap_{n=0}^{\infty} \Delta^{(n)} = \{ z \in \mathbb{R} \}. \)

Since \( f \) has C-der. \( f'(z) \), \( \forall z \in \mathbb{R} \), s.t.

\[ |f(z) - f(z_0) - f'(z_0)(z-z_0)| < \varepsilon |z-z_0|, \quad |z-z_0| < \delta. \]

Now, \( h(z) = f(z) + f'(z_0)(z-z_0) \) is anal. (linear in \( z \)) \Rightarrow

\[ \sum_{k=1}^{4} \frac{1}{T_k} \int \frac{f}{T_k} \, dz = 4 \int \frac{1}{T} \left[ f(z) - f(z_0) - f'(z_0)(z-z_0) \right] \, dz \Rightarrow \]

\[ \left| \frac{1}{T} \int f \, dz \right| \leq 4 \int \frac{1}{T} \left| f(z) - f(z_0) - f'(z_0)(z-z_0) \right| \, dz \leq \varepsilon \text{ diam}(\Delta) \leq 2^{-n} \text{ diam}(\Delta^{(n)}) \leq 2^{-n} \text{ diam}(\Delta) \leq \varepsilon \text{ diam}(\Delta) l(T). \]

Since \( \varepsilon \) arbitrary \( \Rightarrow \) \( \sum_{k=1}^{4} \frac{1}{T_k} \int \frac{f}{T_k} \, dz = 0 \Rightarrow f \) anal. by Morera. \( \square \)