Recall: \( f : (x, y) \rightarrow (\Omega, \rho) \) is cont. if \( \forall a, \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \rho(f(a), f(a)) < \varepsilon \)
when \( d(x, a) < \delta \).

- Equivalently, \( \forall \Delta \subseteq \Omega \text{ open } \Rightarrow f^{-1}(\Delta) \subseteq \Omega \text{ is open.} \)

**Basic Props.**
1. If \( f, g : X \rightarrow \Omega \) are cont. \( \Rightarrow \) \( f\cdot g \), \( f\cdot g \) are cont.
2. \( f : X \rightarrow \Omega \), \( g : \overline{X} \rightarrow \Omega \) are cont. \( \Rightarrow \) \( g \circ f : X \rightarrow \Omega \) is cont.

**Pf** left as Ex. (for 2, use 2nd char. of continuity.)

**Def** 1. \( f : X \rightarrow \Omega \) is uniformly continuous if \( \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \rho(f(x), f(y)) < \varepsilon \text{ when } d(x, y) < \delta. \)

2. \( f \) is Lipshitz cont. if \( \exists C > 0 \text{ s.t. } \rho(f(x), f(y)) \leq C \cdot d(x, y). \)

Clearly: \( f \) Lipshitz \( \Rightarrow \) uniform cont. \( \Rightarrow f \) cont.

**Important Ex.** Let \( A \subseteq \Omega \), and define \( d(\cdot, A) : X \rightarrow \mathbb{R}_+ \) by

\[
d(x, A) := \inf_{y \in A} d(x, y).
\]

Then, \( d(\cdot, A) \) is Lipshitz w/ \( C = 1. \)

**Pf.** Pick \( \varepsilon > 0 \), \( \exists a \in A \text{ s.t. } d(x, a) < d(x, A) + \varepsilon \)

\[
d(y, A) - d(x, A) = d(y, a) - (d(x, a) - \varepsilon) \leq d(y, x) + d(x, a) - d(x, a) + \varepsilon = d(y, x) + \varepsilon.
\]

\( \Rightarrow \) \( d(y, A) - d(x, A) \leq d(x, y) \) since \( \varepsilon \text{ arbitrary.} \)

Thus, \( d(\cdot, A) \) is Lipshitz cont.

(i) If \( K \subseteq \Omega \text{ is compact } \Rightarrow f(K) \subseteq \Omega \text{ is compact.} \)

(ii) If \( A \subseteq \Omega \text{ is connected } \Rightarrow f(A) \subseteq \Omega \text{ is connected.} \)

**Pf.**

(i) Let \( \left\{ \Delta_k \right\}_{k \in I} \) be an open cover of \( f(K) \). Then \( f^{-1}(\Delta_k)_{k \in I} \) is an open cover of \( K \). By assumption, \( \exists \text{ finite subcover } K \subseteq \bigcup_{k=1}^{n} f^{-1}(\Delta_k) \).

But then \( f(K) \subseteq \bigcup_{k=1}^{n} \Delta_k \) finite subcover \( \Rightarrow f(K) \) compact.
But then \( f(K) \leq \bigcup_{k=1}^{\infty} \Delta_{x_k} \), finite subcover \( \Rightarrow f(K) \) compact.

(ii) Suppose \( f(A) \) not connected. \( \Rightarrow \exists B \subseteq \mathbb{R} \) open + closed s.t. \( B \cap f(A) \neq \emptyset \), \( f(A) \). But then \( f^{-1}(B) \subseteq \mathbb{R} \) is open + closed by cont. \( \{ f^{-1}(B) \cap A \neq \emptyset \text{ and } f^{-1}(B) \cap A \neq A \} \).

This is \( \emptyset \) since \( A \) is connected. \( \Box \)

**Important consequence:**

**Theorem 2** Iff \( f: \mathbb{R} \rightarrow \mathbb{R} \) is cont., \( K \subseteq \mathbb{R} \) compact, then

\[ \exists x_1, x_2 \in K \text{ s.t.} \]

\[ \sup_{x \in K} f(x) = f(x_1) \quad \text{and} \quad \inf_{x \in K} f(x) = f(x_2) \]

**Proof:** By Thm 1, \( f(K) \subseteq \mathbb{R} \) is compact. By Heine-Borel, \( f(K) \) is closed and bounded. \( \Rightarrow Y_1 = \inf_{y \in f(K)} y \), \( Y_2 = \sup_{y \in f(K)} y \)

belong to \( f(K) \) \( \Rightarrow \exists x_1, x_2 \in K \text{ s.t.} f(x_1) = Y_1 \), \( f(x_2) = Y_2 \). \( \Box \)

**Very important result:**

**Theorem 3.** Iff \( f: \mathbb{R} \rightarrow \mathbb{R} \) cont. and \( X \) compact \( \Rightarrow f \) is unif. cont.

**Proof:** Pick \( \varepsilon > 0 \). For every \( a \in X \exists \delta_a > 0 \text{ s.t.} |f(x) - f(a)| < \frac{\varepsilon}{2} \)

when \( d(x, a) < \delta_a \). Consider \( G_a = B(a, \delta_a) \) open. If

\[ x, y \in G_a \Rightarrow \rho(f(x), f(y)) \leq \rho(f(x), f(a)) + \rho(f(a), f(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \]

Now \( \{G_a\}_{a \in X} \) is an open cover of \( X \). Since \( X \) compact \( \Rightarrow \) seq. compact, by Lebesgue’s Covering Lemma \( \exists \delta > 0 \)

\[ \forall b \in X \exists a \in X \text{ s.t.} B(b, \delta) \subseteq G_a = B(a, \delta_a) \text{ for some } a. \text{ But then} \]

\[ d(x, b) < \delta \Rightarrow x, b \in G_a \Rightarrow \rho(f(x), f(b)) < \varepsilon. \Rightarrow \]
\[ \text{If } d(x, b) < \varepsilon \Rightarrow x, b \in G_a \Rightarrow C(f(x), f(b)) < \varepsilon. \Rightarrow \]

\text{Thm 4. If } F \subseteq X \text{ closed, } K \subseteq X \text{ compact, } F \cap K = \emptyset \Rightarrow d(F, K) = \inf_{x \in F} d(x, y) > 0. \]

\text{Pr. Observe } d(F, K) = \inf_{x \in F} d(x, F). \text{ Now, } f : X \to \Omega \text{ given by } f(x) = d(x, F) \text{ is cont. (Lipschitz). Since } K \text{ compact, Thm 2 } \Rightarrow \exists x_0 \in K \text{ s.t. } f(x_0) = d(x_0, F) = \inf_{x \in F} d(x, F) = d(F, K). \]

But it is easy to see that \( d(x_0, F) = 0 \Leftrightarrow x_0 \) \text{ a limit point of } F. \text{ Since } F \text{ is closed, if } x_0 \text{ were limit pt of } F \text{ then } x_0 \in F. \text{ But } F \cap K = \emptyset \Rightarrow d(x_0, F) > 0. \]