Chapter 3: Elementary properties and examples of analytic functions

§ 1 Power Series.

Def. If \( a_n \) is in \( \mathbb{C} \) for every \( n \geq 0 \), then the series converges to \( z \in \mathbb{C} \)

\[ \iff \quad \forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \text{ s.t. } \left| \sum_{n=0}^{m} a_n - z \right| < \varepsilon \quad \text{whenever} \quad m > N. \]

Accordingly, we denote \( z = \sum_{n=0}^{\infty} a_n = \lim_{m \to \infty} S_m \).

The series converges absolutely if \( \sum |a_n| \) converges.

Prop 11. If \( \sum |a_n| \) converges absolutely, then \( \sum a_n \) converges.

Pf. Let \( \varepsilon > 0 \) and set \( z = a_0 + a_1 + \cdots + a_n \).

Since \( \sum |a_n| \) converges,

\[ \exists N \in \mathbb{N} \text{ s.t. } \left| \sum_{n=0}^{m} |a_n| - \sum_{n=0}^{m} |a_n| \right| < \varepsilon \]

\[ = \frac{\varepsilon}{n+1} |a_n|. \]

Thus, whenever \( m > N \),

\[ |z_m - z_n| = \left| \sum_{n=0}^{m} a_n - \sum_{n=0}^{m} a_n \right| \leq \sum_{n=N+1}^{m} |a_n| \leq \frac{\varepsilon}{n+1} |a_n| < \varepsilon. \]

That is, \( \{z_n\} \) is a Cauchy sequence and so,

\[ \exists z \in \mathbb{C} \text{ s.t. } z = \lim z_n. \]

\[ \text{Hence, } z = \sum_{n=0}^{\infty} a_n. \]

Def. Let \( \{a_n\} \) be a sequence in \( \mathbb{R} \).

Define \( \lim \inf a_n = \lim \inf \{a_n, a_{n+1}, \ldots\} \)

\[ \limsup a_n = \limsup \{a_n, a_{n+1}, \ldots\} \]

Sometimes we also write

\[ \lim \inf a_n = \inf_{n \to \infty} a_n \]

\[ \lim \sup a_n = \sup_{n \to \infty} a_n \]
Prop. If \( \{a_n\} \) is a convergent sequence in \( \mathbb{R} \) and \( a = \lim a_n \),
then \( a = \liminf a_n = \limsup a_n \).

Prop. \( \liminf a_n \leq \limsup a_n \) for any sequence in \( \mathbb{R} \).

A power series about \( a \) is an infinite series of the form \( \sum_{n=0}^{\infty} a_n (z-a)^n \).

Ex. If \( |z| < 1 \), then
\[
\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.
\]
\[
\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1-z} \quad \rightarrow \quad \frac{1}{1-z}
\]
\( |z| < 1 \Rightarrow |z|^{n+1} \rightarrow 0 \)

Thm. 1.3. For a given power series \( \sum_{n=0}^{\infty} a_n (z-a)^n \),
define the number \( R \), \( 0 \leq R < \infty \), by
\[
\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}}
\]
then (a) if \( |z-a| < R \),
then the series converges absolutely.
(b) if \( |z-a| > R \),
then the terms of the series become unbounded
and so the series diverges.
(c) if \( 0 < R < \infty \),
then the series converges uniformly on \( \{ z : |z-a| \leq R \} \).

Moreover, the number \( R \) is the only number
having property (a) and (b).
pf. We may suppose $a = 0$.

(a) If $|z| < R$, there is an $r$ with $|z| < r < R$.

Thus, $\exists N \in \mathbb{N}$, s.t. $|a_n| < \frac{1}{n^k}$ for all $n \geq N$. $\frac{1}{n^k} > \frac{1}{r^k}$

Then $|a_n| < (\frac{1}{r})^k$ and $|a_n z^n| < (\frac{|z|}{r})^k$.

Thus $\sum_{n=N}^{\infty} a_n z^n$ is dominated by $\frac{|z|^k}{r^k} (\frac{1}{r})^k$

\[\frac{|z|^k}{r^k} < 1, \quad \frac{\infty}{n=0} (\frac{|z|^k}{r^k})^n\]

Thus, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.

(b) Exercise

(c) Fix $r < |z|$, and choose $r < f < R$

$\exists N$, s.t. $|a_n| < \frac{1}{n^k}$ for $n \geq N$, as above.

If $|z| = r$, $|a_n z^n| < \frac{1}{n^k}$ for $n \geq N$.

$\sum_{n=0}^{\infty} a_n z^n$ is dominated by $\frac{\infty}{n=0} (\frac{1}{r})^n$

Hence $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on $|z|: |z| \leq r$.

Remark. $R$ is called the radius of convergence of the power series.

Prop 1.4. If $\sum a_n (z-a)^n$ is a given power series with radius of convergence $R$, then

$R = \lim a_n$ if the limit exists.

Proof. We may assume $a = 0$.

Let $d = \lim a_n$.

Suppose $|z| < \alpha$. WTS $\sum |a_n z^n|$ converges.

Take $r$ s.t. $|z| < r < \alpha$.

$\exists N > 0$, s.t. $|a_n| > r$ for $n > N$. 


Let $B = |a_n| \cdot r^N$

Then

$|a_{N+1} \cdot r^{N+1}| \leq \frac{|a_n|}{r} \cdot r^{N+1} = |a_n| \cdot r^N = B$

$|a_{N+2} \cdot r^{N+2}| \leq \frac{|a_{N+1}|}{r} \cdot r^{N+2} = |a_{N+1}| \cdot r^{N+1} = B$

Inductively, $|a_n r^n| \leq B$ for $n > N$.

Since $|2| < r$, $|a_n z^n| = |a_n r^n| \cdot |\frac{z}{r}|^n \leq B \left(\frac{|z|}{r}\right)^n \to n \geq N$.

So $\sum |a_n z^n|$ converges.

Suppose $|2| > 2$.

Then $\sum a_n z^n$ diverges.

Take $r$ s.t. $2 < r < |2|$.

Since $2 = \lim \frac{|a_n|}{|a_{N+1}|} \leq r$,

$2N > 0$ s.t. $\frac{|a_n|}{|a_{N+1}|} < r$ for $n > N$.

Set $B = \lim |a_n r^N| > 0$.

Then

$|a_{N+1} r^{N+1}| > \frac{|a_n|}{r} \cdot r^{N+1} = |a_n| \cdot r^N = B$

$|a_{N+2} r^{N+2}| > \frac{|a_{N+1}|}{r} \cdot r^{N+2} = |a_{N+1}| \cdot r^{N+1} = B$.

Inductively, $|a_n r^n| > B$ for any $n > N$.

$|a_n z^n| > |a_n r^n| > B \to 0$ as $n \to \infty$.

So $\sum a_n z^n$ diverges.

Therefore $R = 2$.

Ex. Consider $\frac{\infty}{n^2, \frac{n^2}{n^2}}$

Note $a_n = \frac{1}{n!}$, $\frac{a_n}{a_{N+1}} = n+1 \to +\infty$.

So $R = +\infty$

$e^z \overset{\text{def}}{=} \lim_{n \to \infty} \frac{z^n}{n!}$
Prop. 1.5. Let \( \sum a_n \) and \( \sum b_n \) be two absolutely convergent series and put 
\[
C_n = \frac{2}{n} \cdot a_n \cdot b_n
\]

Then \( \sum C_n \) is absolutely convergent with the sum
\[
\sum C_n = \sum a_n \cdot \sum b_n
\]

Prop. 1.6. Let \( \sum a_n (z-a)^n \) and \( \sum b_n (z-a)^n \) be power series with

radius of convergence \( r > 0 \).

Put \( C_n = \frac{n}{n+1} \cdot a_n \cdot b_n \).

Then both power series \( \sum (a_n + b_n) (z-a)^n \) and \( \sum C_n (z-a)^n \)

have radius of convergence \( r > 0 \), and
\[
\sum (a_n + b_n) (z-a)^n = \sum a_n (z-a)^n + \sum b_n (z-a)^n
\]
\[
\sum C_n (z-a)^n = \sum a_n (z-a)^n \cdot \sum b_n (z-a)^n
\]

for \( |z-a| < r \)

5.2. Analytic functions.

Def. 2.1. If \( G \) is an open set in \( \mathbb{C} \) and \( f : G \to \mathbb{C} \),

then \( f \) is differentiable at a point \( a \in G \) if
\[
 f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists} \quad \text{in} \, \mathbb{C}
\]

\( f'(a) \) will be called the (complex) derivative of \( f \) at \( a \).

1. If \( f \) is differentiable at each point of \( G \), we say \( f \) is differentiable on \( G \).

\( f' : G \to \mathbb{C} \)

If \( f' \) is continuous, then we say \( f \) is continuously differentiable.

2. If \( f' \) is (complex) differentiable, we say \( f \) is twice differentiable.

Inductively, we can define \( f \) is infinitely differentiable.
Def 2.2. If \( f : \mathbb{C} \to \mathbb{C} \) is differentiable at \( a \in \mathbb{C} \), then \( f \) is continuous at \( a \).

Def 2.3. A function \( f : \mathbb{C} \to \mathbb{C} \) is analytic if \( f \) is continuously (complex) differentiable.

Chain Rule 2.4. Let \( f, g \) be analytic on \( G \) and \( N \) respectively. Suppose \( f(z) \equiv a \).

Then \( g \circ f \) is analytic in \( G \), and
\[
(g \circ f)'(z) = g'(f(z)) \cdot f'(z) \quad \text{for any } z \in G.
\]

Remark. We have defined analytic function \( f \) on open set \( G \).

For any arbitrary set \( A \), \( f \) is analytic on \( A \) if \( f \) is analytic in some open set \( G \) and \( A \subseteq G \).

Example. \( f : \mathbb{C} \to \mathbb{C} \), \( f(z) = \bar{z} \).

\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \lim_{h \to 0} \frac{\bar{h}}{h} = \lim_{h \to 0} \frac{1}{\bar{h}} \text{ if } h \in \mathbb{R}.
\]

\[
\frac{1}{\bar{h}} = \begin{cases} 
1 & \text{if } h \in \mathbb{R} \\
-1 & \text{if } h \in \mathbb{H}
\end{cases}
\]

Thus, \( f \) is not (complex) differentiable.

\( f(x + iy) = x - iy \) is differentiable w.r.t. real variables \( x \) and \( y \).
Prop 2.5 Let \( f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \) have radius of convergence \( R > 0 \).

Then,

a) For each \( k \geq 1 \), the series
\[
\sum_{n=k}^{\infty} \frac{b_{n-k}}{(n-k)!} \frac{\partial^k f(z)}{\partial z^k} = \frac{b_{n-k}}{(n-k)!} \frac{\partial^k f(z)}{\partial z^k}
\]
has radius of convergence \( R \).

b) The function \( f \) is indefinitely differentiable in \( B(a, R) \) and furthermore, \( f^{(k)}(z) \) is given by (4) for any \( k \geq 1 \) and \( |z-a| < R \).

c) For \( n \geq 0 \), \( a_n = \frac{1}{n!} f^{(n)}(a) \).

proof. Again assume that \( a = 0 \).

(a) It suffices to prove for \( k = 1 \).

Recall \( R^1 = \limsup |a_n|^{\frac{1}{n}} \).

WTS: \( R^1 = \limsup |n a_n|^{\frac{1}{n}} \).

Sufficient to check \( \lim_{n \to \infty} \frac{n^{\frac{1}{n}}}{a_n} = 1 \)

\[
\lim_{n \to \infty} \frac{n^{\frac{1}{n}}}{a_n} = \lim_{n \to \infty} \frac{1}{n^{\frac{1}{n}}} = 0 \quad \text{as} \quad n \to \infty
\]

b) We first prove for \( k = 1 \).

For \( |z| < R \), put \( g(z) = \frac{1}{1!} \frac{\partial f(z)}{\partial z} \).

\[
S_n(z) = \sum_{j=0}^{n} a_j z^j, \quad R_n(z) = \sum_{j=n+1}^{\infty} a_j z^j
\]

Fix \( w \in B(0, R) \) and take \( r \) s.t. \( |w| < r < R \).

WTS: \( f'(w) = g(w) \).
Take $s > 0$ s.t. $\overline{B(w, s)} \subseteq B(0, r)$

Let $z \in B(w, s)$

\[
\frac{f(z) - f(w)}{z - w} = g(w)
\]

\[
= \left( \frac{s_n(z) - s_n(w)}{z - w} - s_n'(w) \right) + \left( s_n'(w) - g(w) \right) + \left( \frac{R_n(z) - R_n(w)}{z - w} \right)
\]

\[
\frac{R_n(z) - R_n(w)}{z - w} = \frac{\sum_{j=1}^{\infty} a_j (z^{j-1} w)}{z - w}
\]

\[
= \frac{\sum_{j=1}^{\infty} a_j (z^{j-1} + z^{j-2} w + \ldots + w^{j-1})}{z - w}
\]

\[
\left| \frac{R_n(z) - R_n(w)}{z - w} \right| \leq \frac{\sum_{j=1}^{\infty} |a_j| |z^{j-1} + z^{j-2} w + \ldots + w^{j-1}|}{|z - w|}
\]

\[
\leq \frac{\sum_{j=1}^{\infty} |a_j| (r^{j-1} + r^{j-2} + \ldots + r^{j-1})}{|r - 1|}
\]

\[
= \frac{\sum_{j=1}^{\infty} |a_j| \cdot r^{j-1}}{|r - 1|}.
\]

Since $R < R^*$, \( \sum_{j=1}^{\infty} |a_j| \cdot r^{j-1} < +\infty \)

Thus $\exists N_1 \in \mathbb{N}$ s.t. $\sum_{j=1}^{\infty} |a_j| \cdot r^{j-1} < \frac{\varepsilon}{3}$ for $n > N_1$.

\[
S_n(z) = \sum_{j=1}^{n} a_j \cdot w^{j-1} \rightarrow \sum_{j=1}^{\infty} i a_j \cdot w^{j-1} = g(w) \quad \text{as} \quad n \rightarrow \infty.
\]

$\exists N_2 \in \mathbb{N}$ s.t. $\left| s_n'(w) - g(w) \right| < \frac{\varepsilon}{3}$ for $n > N_2$. 

\[
\exists N \in \mathbb{N} \quad \text{s.t.} \quad \left| s_n'(w) - g(w) \right| < \frac{\varepsilon}{3} \quad \text{for} \quad n > N.
\]
Take $n = \max (N_1, N_2)$. Then

$|S_n(w) - G(w)| < \varepsilon$

$\left| \frac{R_n(z) - R_n(w)}{z - w} \right| < \varepsilon$

$S_n(z) = \lim_{n \to w} \left( \frac{S_n(z) - S_n(w)}{z - w} - S_n'(w) \right) = 0$ for $n = \max (N_1, N_2)$,

$\exists \delta > 0$, s.t. $\left| \frac{S_n(z) - S_n(w)}{z - w} - S_n'(w) \right| < \varepsilon$ for $|z - w| < \delta$.

So $\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| < \varepsilon$ for $|z - w| < \delta$.

That is

$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w} = g(w)$.

$c)

f^{(k)}(z) = \frac{n!}{n-k} \frac{n(n-1) \ldots (n-k+1)}{n!} a_{n-k} z^{n-k}.$  by (a)

$f^{(k)}(z) = k! a_k.$

Cor. 29 If the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence $R > 0$,

then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ is analytic in $B(a, R)$.

Ex. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is analytic in $\mathbb{C}$.

$$(e^z)' = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z$$

$\quad e^{a+b} = e^a e^b$

$e^z e^{-z} = e^0 = 1$

$\overline{e^z} = e^{\overline{z}}$

$|e^z| = |e^z \overline{e^z}|^{1/2} = |e^{2i\pi /2}| = e^{\Re z}$
Def. For $z \in \mathbb{C}$, define
\[ \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} \]
\[ \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n-1}}{(2n-1)!} \]

RMK.
- Convergence radius $R = +\infty$
- $\cos, \sin$ are extensions of $\cos x, \sin x$ for $x \in \mathbb{R}$

Ref.
\[ \cos z = \frac{1}{2} (e^{iz} + e^{-iz}) \]
\[ \sin z = \frac{1}{2i} (e^{iz} - e^{-iz}) \]
\[ e^{iz} = \cos z + i \sin z \]
\[ \cos^2 z + \sin^2 z = 1 \]
\[ z = |z| e^{i \theta} \quad \theta = \arg z \]

Def. A function $f$ is periodic with period $c$ if $f(z+c) = f(z)$ for all $z \in \mathbb{C}$

Ex. Find the period of $e^z$.
\[ e^{z+c} = e^z \quad e^c = 1 \]
\[ c = a + ib \quad a, b \in \mathbb{R} \]
\[ e^c = e^a (\cos b + i \sin b) = 1 \]
\[ a = 0 \quad b = 2k\pi \quad \text{for } k \in \mathbb{Z} \]
\[ c = 2\pi ki \quad k \in \mathbb{Z} \]

$z \rightarrow e^z : \mathbb{C} \rightarrow \mathbb{C}$ is not one-to-one.

It has no inverse functions.

We take a branch of its multi-valued inverse.
Def. If \( G \) is an open and connected subset of \( \mathbb{C} \),
and \( f : G \to \mathbb{C} \) is a continuous function
\( \text{s.t. } z = \exp f(z) \text{ for } z \in G \),
then \( f \) is a branch of the logarithm.

Rmk. \[ |e^z| = e^{\text{Re}z} > 0. \]
\[ e^z : \mathbb{C} \to \mathbb{C} - \{0\}. \]
So we must have \( 0 \notin G \).

Prop. If \( G \subseteq \mathbb{C} \) is open and connected, and \( f : G \to \mathbb{C} \) is a
branch of \( \log z \),
then \( g : G \to \mathbb{C} \) is a branch of \( \log z \),
\[ \text{iff } g(z) = f(z) + 2k\pi i \text{ for some } k \in \mathbb{Z}. \]

Proof. ('\( \Rightarrow \)') easy.
\[ \Rightarrow \text{ set } h(z) = \frac{1}{2\pi i} (f(z) - g(z)) \]
\[ e^{2\pi i} h(z) = e^{f(z)} - e^{g(z)} \]
\[ = \frac{z}{z} \]
\[ = 1 \text{ for } z \in G \]
Thus \( h(z) \in \mathbb{Z} \).

Since \( h \) is continuous, and \( G \) is connected,
\( h \) is a constant integer.
So \( f(z) - g(z) = 2\pi i k \) for some \( k \in \mathbb{Z}. \)

Ex. \[ G = \mathbb{C} - \{ z \in \mathbb{R} : z \leq 0 \} \]
\[ f : G \to \mathbb{C} \]
\[ \forall z \in G, z = re^{i\theta} \theta \in (-\pi, \pi) \]
\[ f(z) = f(re^{i\theta}) \text{ defines } \log r + i\theta. \text{ principle branch.} \]
Prop. Let \( G \) and \( \mathbb{R} \) be open subsets of \( \mathbb{C} \).

Suppose that \( f: G \to \mathbb{C} \) and \( g: \mathbb{R} \to \mathbb{C} \) are continuous functions

\[
s.t. \quad f(z) \in \mathbb{R} \quad \text{and} \quad g(f(z)) = z \quad \text{for} \quad z \in \mathbb{R}.
\]

If \( g \) is differentiable and \( g'(z) \neq 0 \),

then \( f \) is differentiable and

\[
f'(z) = \frac{1}{g'(f(z))}.
\]

Furthermore, if \( g \) is analytic, then \( f \) is analytic.

If: Fix \( a \in G \), and let \( h \in \mathbb{C} \) s.t. \( h \neq 0 \) and \( a + h \in G \).

Note that \( \alpha = g(f(a)) \) and \( \alpha + h = g(f(a + h)) \)

implies that \( f(a) \neq f(a + h) \).

\[
1 = \frac{g(f(a + h)) - g(f(a))}{h} = \frac{g(f(a + h)) - g(f(a))}{h} \quad \text{for} \quad h \\
\]

Thus,

\[
\left| \frac{g'(f(a)) - g'(f(a))}{h} \right| \to 0 \quad \text{as} \quad h \to 0
\]

Thus,

\[
\lim_{h \to 0} \frac{g(f(a + h)) - g(f(a))}{f(a + h) - f(a)} = g'(f(a))
\]

Thus, \( h \to 0 \quad f(a + h) - f(a) \quad \text{exists} \)

and

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \frac{1}{g'(f(a))}.
\]

Thus,

\[
f'(a) = \frac{1}{g'(f(a))} \quad \text{is continuous}
\]

So, \( f \) is analytic.

RMK: Any branch of \( \log \) is analytic.
Complex - Riemann equation

\[ f: \mathbb{C} \rightarrow \mathbb{C} \]

\[ z \rightarrow f(z) \]

\[ z = x + iy \quad f(x+iy) = u(x,y) + iv(x,y) \]

Suppose \( f \) is analytic

\[ f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \]

We evaluate the limit in two ways.

1. Take \( h \in \mathbb{R} \)

\[ \frac{f(z+h) - f(z)}{h} = \frac{u(x+th, y) - u(x,y)}{h} + i \frac{v(x+th, y) - v(x,y)}{h} \]

\[ \rightarrow u_x(x,y) + iv_x(x,y) \]

2. Take \( h = it \) and \( t \in \mathbb{R} \)

\[ \frac{f(z+ith) - f(z)}{ith} = \frac{u(x, th+y) - u(x,y)}{ith} + i \frac{v(x, th+y) - v(x,y)}{ith} \]

\[ \rightarrow -i u_y(x,y) + v_y(x,y) \]

Comparing the limits,

\[ \begin{align*}
    u_x(x,y) &= v_y(x,y) \\
    u_y(x,y) &= -v_x(x,y)
\end{align*} \]

This is the so-called Cauchy-Riemann equations.

Prop 2.10: \( G \) is a region in \( \mathbb{C} \)

\( f: G \rightarrow \mathbb{C} \) is analytic and \( f' \equiv 0 \) in \( G \)

Then \( f \) is a constant.
\( f : \mathbb{C} \to \mathbb{C} \)

\[ f = u + iv \]

\( f \) is analytic \( \Rightarrow \) Cauchy–Riemann equations.

Q. How about the converse?

**Thm 2.29.** Let \( u \) and \( v \) be real-valued functions defined on a region \( \mathbb{C} \) and suppose that \( u \) and \( v \) have continuous partial derivatives.

Then \( f : \mathbb{C} \to \mathbb{C} \) defined by \( f(z) = u + iv \) is analytic iff \( u \) and \( v \) satisfy the Cauchy–Riemann equations.

pf. \( \Rightarrow \)

\[ f = u + iv \]

\[ \frac{f(x+it) - f(x)}{it} = \frac{U(x_2, y_2) - U(x, y)}{it} + i \frac{V(x_2, y_2) - V(x, y)}{it} \]

\[ u(x_2, y_2) - u(x, y) \]

\[ = u(x_1, y_1) - u(x_1, y_1) + u(x_1, y_1) - u(x, y) \]

\[ = u_x(x_1, y_1) \cdot s + u_y(x_1, y_1) \cdot t \]

\( s \in (0, s) \)

\( t \in (0, t) \)

Similarly,

\[ v(x_2, y_2) - v(x, y) \]

\[ = v_x(x_1, y_1) \cdot s + v_y(x_1, y_1) \cdot t + o(s) + o(t) \]
\[
\frac{f(z+st+it) - f(z)}{st+it} = \frac{u_x(x,y)s + u_y(x,y)t + iv_x(x,y)s + iv_y(x,y)t}{st+it} + o(1)
\]

\[
= \frac{(u_x(x,y) + iv_x(x,y))s + (u_y(x,y) + iv_y(x,y))t}{st+it} + o(1)
\]

CR equations:

\[
\frac{d}{dt}(u_x(x,y) + iv_x(x,y))s + (u_y(x,y) + iv_y(x,y))t + o(1)
\]

\[
= u_x(x,y) + iv_x(x,y) + o(1)
\]

\[
\lim_{st+it \to 0} \frac{f(z+st+it) - f(z)}{st+it} = u_x(x,y) + iv_x(x,y)
\]

\[\Box\]