Recall. Assume $u,v: G \to \mathbb{R}$ have cont. partial derivatives (aka $C^1$) then: $f = u + iv: G \to \mathbb{C}$ is analytic $\iff \begin{cases} U_x = V_y \\
U_y = -V_x \end{cases}$ (Cauchy-Riemann or CR) in $G$.

**Def.** A function $u$ is harmonic if $u \in \mathbb{C}^2$ and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0 \text{ in } G.$$ 

**Rem.** If $u, v$ are $C^2$ and satisfy CR, then since mixed partial derivatives commute

$$u_{xx} = v_{yy} = v_{yx} = -u_{yy},$$

i.e. $u_{xx} + u_{yy} = 0$, so $u$ is harmonic.

A similar computation shows $v$ is harmonic.

**Def.** If $u, v$ are harmonic in $G$ and $f = u + iv$ is analytic (i.e. satisfy CR), then $v$ is harmonic conjugate of $u$.

**Prop.** Assume $v_1, v_2$ are both harmonic conjugates of $u$ in $G$, and $G$ is connected. Then $\exists$ constant $C \in \mathbb{C}$ s.t. $v_1 - v_2 = C$.

**Pf.** Fix $B(a,r) \subseteq G$. For simplicity of notation and WLOG: $a = 0$.

For any harm. conj. $v$, \[ \left\{ \begin{array}{l}
V_x = -U_y \\
V_y = U_x
\end{array} \right. \]

\[ V(x, y) - V(0, 0) = V(x, y) - V(x, 0) + V(x, 0) - V(0, 0) = \int \left[ \int_{x}^{y} V_y(x, t) \, dt + \int_{0}^{y} V_x(s, 0) \, ds \right] 
\]

$\therefore \int \left[ \int_{x}^{y} V_y(x, t) \, dt + \int_{0}^{y} V_x(s, 0) \, ds \right] 
\]

Thus, $V(x, y) = \int_{0}^{y} U_x(x, t) \, dt - \int_{0}^{y} U_y(s, 0) \, ds + V(0, 0)$. 

Therefore: $\int_{0}^{y} U_x(x, t) \, dt - \int_{0}^{y} U_y(s, 0) \, ds + V(0, 0) = C.$
$\Rightarrow$ in $B(a, r) \subseteq G$, $v_1(x+iy) - v_2(x+iy) = v_1(a) - v_2(a) = C$

Let $A := \{z = x+iy \in G : v_1(x+iy) - v_2(x+iy) = C \}$. Then $A \neq \emptyset$ and

Claim 1. A is open.

Let $a \in A$, and choose $B(a, r) \subseteq G$. By same arg. as above

$v_1(x+iy) - v_2(x+iy) = v_1(a) - v_2(a), \forall x+iy \in B(a, r)$. But

$v_1(a) - v_2(a) = C$ by assumption $\Rightarrow B(a, r) \subseteq A \Rightarrow A$ open.

Claim 2. A is closed.

Let $b$ be limit point of $A$ and $\{a_n\}$ seq. in $A$ s.t. $a_n \to b$. Then

$C = v_1(a_n) - v_2(a_n) \to v_1(b) - v_2(b)$ by cont. of $v_1, v_2$.

But then $b \in A \Rightarrow A$ closed since it contains all of its limit points.

Thus 1. Assume $G$ is either $B(a, r)$ or $C$. Then, every harmonic $u : G \to \mathbb{R}$ has a unique harmonic conjugate up to additive constant.

Pf. The part about unique up to additive constant is the content of Prop 1 (for any connected $G$).
The existence follows from the proof of Prop 1.
In $B(a, r)$ or $C$, we can define $v$ by (WLOG, $a = 0$)

$v(x+iy) = \int_0^x u_x(x, t) dt - \int_0^y u_y(s, 0) ds$.

By FTC, $v_y(x+iy) = u_x(x, y)$.

Also, $v_x(x+iy) = \int_0^y u_{xx}(x, t) dt - u_y(x, 0) = \{A u = 0\}$

\[ v_1, v_2, \ldots, v_n, \ldots \]
\[ \frac{d}{dt} \Phi_y(x, t) + \Phi_y(x, t) \frac{dx}{dt} = \Phi_y(x, 0) \]

\[ = -\left( \Phi_y(x(t), t) - \Phi_y(x(0), t) \right) - \Phi_y(x(0), t) \]

\[ = -\Phi_y(x(t)) \]

\[ \frac{\partial}{\partial t} \Phi_y(x, t) \]

**Remark:** If requires \( \exists \theta \in G \) s.t. \( z = x + iy \) then figure is contained in \( G \) for all \( z = x + iy \in G \).

- In \( G = \mathbb{C} \setminus \{0\} \), \( u = \log |z|^2 \) is harmonic (Why?) but
- if \( v \) were harmonic, then \( f = u + iv = \log z + C \), for some analytic branch of \( \log z \) in \( G \), but as we have seen, there is no analytic branch of \( \log z \) in \( \mathbb{C} \setminus \{0\} \).

**Analytic functions as mappings.**

**Def.** A path (or curve) in \( G \subset \mathbb{C} \) (region) is a cont. map \( \gamma : [a, b] \to G \). \( \gamma \) is \((C^1)\) smooth if \( \gamma' \) exists and is cont. on \([a, b]\). \( \gamma \) is piecewise smooth if \([a, b] = \bigcup_{k=1}^{m} [a_k, a_{k+1}] \) with \( a_0 = a \), \( a_n = b \) and \( \gamma \) smooth on each \([a_k, a_{k+1}]\).

(2) If \( \gamma_1, \gamma_2 : [a, b] \to G \) are smooth paths and for some \( t_0 \in (a, b) \)
\( \gamma_1(t_0) = \gamma_2(t_0) = z_0 \), and \( \gamma_1'(t_0) \neq 0 \); \( \gamma_2'(t_0) \neq 0 \), then the angle between \( \gamma_1 \) and \( \gamma_2 \) is \( \angle (\gamma_1, \gamma_2)_{t_0} \in [-\pi, \pi] \).
The angle between \( \gamma_1 \) and \( \gamma_2 \) is \( \angle (\gamma_1, \gamma_2) \equiv \angle (\tilde{\gamma}_1, \tilde{\gamma}_2) \equiv \angle \left( \gamma_1(t_0), \gamma_2(t_0) \right) \).  

*Note on Conway's def.*

**Note:** If \( f : G \to \mathbb{C} \) is a complex function, we can view it as a map:

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{f} & \mathbb{C} \\
\downarrow & & \downarrow \\
G & \xrightarrow{\mu} & f(G)
\end{array}
\]

(or angle preserving)

- If \( f = \text{univ w/ } u, v \in \mathbb{C} \), \( f \) is conformal at \( z_0 \) if \( \forall \) smooth paths \( \gamma_1, \gamma_2 : [a, b] \to G \), as in \( \theta \) above, the angle between \( \gamma_1 \) and \( \gamma_2 \) at \( t_0 \) equals the angle between \( \mu_1 := f \circ \gamma_1 \) and \( \mu_2 := f \circ \gamma_2 \) at \( t_0 \) (meaning also \( \mu_1'(t_0) \neq 0 \), \( \mu_2'(t_0) \neq 0 \)).

**Thm. 2.** If \( f \) is analytic in \( G \) and \( f' \neq 0 \) in \( G' \subseteq G \), then \( f \) is conformal in \( G' \).

**Pf.** Let \( \gamma_1, \gamma_2 \) be as above w/ \( z_0 = \gamma_1(t_0) = \gamma_2(t_0) \) e \( G' \). Since \( f \) is analytic, chain rule \( \Rightarrow \)

\[
\mu_1'(t_0) = f'(z_0) \gamma_1'(t_0) \quad \text{and} \quad \mu_2'(t_0) = f'(z_0) \gamma_2'(t_0)
\]

and \( f'(z_0) \neq 0 \). Thus, \( \mu_1'(t_0), \mu_2'(t_0) \) are obtained from \( \gamma_1'(t_0), \gamma_2'(t_0) \) by multiplication by same
from \( f_1(z), f_2(z) \) by multiplication by same nonzero number \( f(z) \) we immediately see 
\[
\lambda (f_1(z_1)f_2(z_2)) = \lambda (f_1(z_1)f_2(z_2)) \Rightarrow f \text{ is conformal at } z_0.
\]

\[\square\]

**Möbius Transformations.**

**Def.** A **linear fractional transformation** is a map \( S : \mathbb{C} \to \mathbb{C} \)
given by 
\[
S(z) = \frac{az + b}{cz + d}, \quad S(-d/c) = 0, \quad S(\infty) = a/c.
\]

- \( S(z) \) is a **Möbius transformation** if \( ad - bc \neq 0 \).

**Basic Props of Möbius Trans.**

1. \( S : \mathbb{C} \to \mathbb{C} \) is a homeomorphism (cont. & bijective), and
   an analytic function on \( \mathbb{C} - \{ -d/c \} \).

2. The inverse is 
   \[
   S^{-1}(z) = \frac{d_2 - b}{-cz + a}.
   \]

3. If \( S(z) = z \) has 3 solutions \( z_1, z_2, z_3 \in \mathbb{C} \) (fixed pts),
   then in fact \( S(z) \) is identity map (\( S(z) = z \)).

**Sketch of pf.**

1. Analyticity is clear. Cont. on \( \mathbb{C} \) is Ex.
   Bijection follows from (2).

2. Solve equation \( w = \frac{az + b}{cz + d} \) for \( z \).

3. Consider the equation
   \[
   z = \frac{az + b}{cz + d} \Rightarrow cZ^2 + (d - a)Z - b = 0. \quad (x)
   \]
   Either \( c = 0, \ d = a, \ b = 0 \) \( \Rightarrow S(z) = z \) or
   (no solution).

Either \( c = 0, \ d = a, \ b = 0 \) \( \Rightarrow S(z) = z \) or

\[\square\]
Either $c = 0$, $\alpha = \infty$, $a - c = \infty$, or

(x) is non-trivial quadratic eq. $\Rightarrow$ can at most have 2 distinct solutions.

Prop 2. Given $Z_1, Z_2, Z_3 \in \mathbb{C}$, unique Möbius $S$ w/ $S(Z_1) = 1$, $S(Z_2) = 0$, $S(Z_3) = \infty$.

Let $Z_1, Z_2, Z_3 \in \mathbb{C}$ then

$$S(Z) = \frac{Z-Z_2}{Z-Z_3} \cdot \frac{Z_1-Z_3}{Z_1-Z_2}.$$ 

If say $Z_2 = \infty$, then $S(Z) = \frac{Z_1-Z_3}{Z-Z_3}$, etc.

Thus, given $Z_1, Z_2, Z_3, W_1, W_2, W_3 \in \mathbb{C}$, unique $S$ w/ $S(Z_j) = W_j$ for $j = 1, 2, 3$.

Let $T_1, T_2$ be maps given by Prop 2 for $(Z_1, Z_2, Z_3)$ and $(W_1, W_2, W_3)$ respectively. Then $S = T_2^{-1} \circ T_1$ does the trick. To see this, $S$ is unique, we note that if there are $S_1, S_2$ that sends $(Z_1, Z_2, Z_3)$ to $(W_1, W_2, W_3)$, then $S_2^{-1} \circ S_1$ has 3 fixed points (namely $Z_1, Z_2, Z_3$) and hence, $S_2^{-1} \circ S_1 = I$ or

$S_1 = S_2$. 

$\Box$