Proof of Lemma 1., Cor. 1-2 in Lecture 9 Notes.

Recall: Thm 1. Let \( \Omega \subseteq \mathbb{C}^n \). TFAE:

(i) \( \Omega \) is a d.o. holom.

(ii) \( \forall K \subseteq \Omega, \overline{K} \subseteq \Omega \).

(iii) \( \exists \Omega' \subseteq \Omega \) that does not extend across any body pt. i.e.

\[ \exists \Omega', \quad \forall \Omega, \Omega \cap \Omega' \neq \emptyset, \quad \Omega \cap \Omega' \neq \emptyset, \quad \text{Fe} \Omega' \subseteq \Omega \setminus F_{\Omega} = \emptyset. \]

Proof.

(i) \( \Leftrightarrow \) (iii) is Cor. 2 from Lecture 9.

(iii) \( \Leftrightarrow \) (i) is immediate from def. of d.o. holom.

(ii) \( \Rightarrow \) (iii).

WL06, assume \( \Omega \) connected. Let \( D^n = \{ z \in \mathbb{C}^n : |z| < 1 \} \),

\[ \Delta(z) = \Delta_{D^n}(z) = \sup \{ r > 0 : \{ z \in \mathbb{C}^n : |z| < r \} \subseteq \Omega \} \]

as in Lemma 1, and let \( D_z = \{ z \} + \Delta(z) D^n \subseteq \Omega \). Thus, \( D_z \) is largest polydisk of “shape” \( D^n \) that is centered at \( z \) and contained in \( \Omega \).

Let \( M \) be a countable dense subset of \( \Omega \). Suffices to construct \( \text{Fe} \Omega'(\Omega) \) s.t. \( \Omega \) does not extend to open neighborhood of \( \overline{D}_z \) for any \( z \in M \). For, if \( \Omega_1, \Omega_2 \) as in (iii) exist then \( \exists z \in M \)

\[ \overline{D}_z \subseteq \Omega_2. \]

Now let \( \{ z_j \}_{j=1}^{\infty} \) be a sequence in \( \Omega \) s.t. each \( z \) in \( M \) appears \( \infty \) many times. Let \( \{ K_j \}_{j=1}^{\infty} \) be an exhaustion of \( \Omega \) by compact sets ( \( K_1 \subseteq K_2 \subseteq \ldots \subseteq \Omega = \bigcup_{j=1}^{\infty} K_j \), and \( \forall K \subseteq \Omega, \ K \subseteq K_j, j > 1 \)).

Since \( \hat{K}_j = \bigcup_{j=1}^{\infty} K_j \subseteq \Omega \), by assumption and \( \overline{D}_z \cap \Omega \neq \emptyset \).
Since \( R_j = (K_i)_j \subseteq \Omega \) by assumption and \( D_3 \cap (2\Omega) \neq \emptyset \) by construction, \( \exists Z_j \in D_3 \setminus R_j \). Hence, \( \exists f_j \in C(\Omega) \) s.t. \( f_j(Z_j) = 1 \) and \( \sup_{K_j} |f_j| < 1 \). By replacing \( f_j \) by \( f_j^p \), \( p > 1 \), WLOG assume \( f_j(Z_j) = 1 \), \( \sup_{K_j} |f_j| < \frac{1}{2^j} \). Consider the \( \infty \) product:

\[
 f = \prod_{j=1}^{\infty} (1 - f_j)^{-j}. \tag{1}
\]

We know (Math 220B or C) that this product converges to a holomorphic function in \( \Omega \) s.t. \( f(z) = 0 \) only when \( f_j(z) = 1 \), some \( j \)

\[ \iff \forall K \subseteq \Omega, \sum_{j=1}^{\infty} j \sup_{K_j} |f_j| < \infty \]

But \( K \subseteq K_j, j \geq N \). Since \( \sup_{K_j} |f_j| < \frac{1}{2^j} \) and \( \sum_{j=1}^{\infty} \frac{j}{2^j} < \infty \),

it follows that (1) converges a holomorphic function \( f \) s.t. \( f \neq 0 \) and \( f_{2^k}(z_j) = 0 \), \( \forall |z| < j \). Pick \( z \in M \), \( \exists \) subsequence \( z_{3^k} = z \) and hence \( z_{3^k} \in D_3 \setminus R_j \). Going to a subsequence if necessary.

WLOG assume \( z_{3^k} \to z_0 \in \partial D_3 \). Since \( f \) vanishes to order at least \( 3^k - 1 \) at \( z_{3^k} \), if \( f \) extended as holomorphic function in nbhd of \( D_3 \), we would have \( f_{2^k}(z_0) = 0 \), \( \forall k \implies f \equiv 0 \) in \( D_3 \) and hence in \( \Omega \). Since \( f \neq 0 \) by construction, the pf is complete. \( \blacksquare \)