Pseudoconvexity.

Recall. If \( \delta(z) = \max(\|z\|, \ldots, \|z\|) \) and \( \delta(z, \mathbb{C}^n \setminus \Omega) = \inf_{w \in \Omega} \delta(z, w) \), then:

**Prop 1.** If \( \Omega \subset \mathbb{C}^n \) is d.o.holom., \( \mathbb{K} \subset \mathbb{C} \), \( f \in \mathcal{O}(\Omega) \) s.t. \( f(z) \leq \delta(z, \mathbb{C}^n \setminus \Omega) \)
on K, then \( \|f(z)\| \leq \delta(z, \mathbb{C}^n \setminus \Omega) \) in \( K \).

**Rem.** The conclusion can be proved w/ any continuous function \( \delta(z) \) s.t.

\[ \delta(z) = |11^{\delta(z)}|, \quad \delta(z) = \left( \sum_{i=1}^{\infty} |z_i|^i \right)^{1/p}, \quad \text{\( \ell^p \)-norm}. \]

**Def.** Let \( \Omega \subset \mathbb{C}^n \) d.o.holom. and \( \delta(z) > 0 \) as in Rem above.

Then, the log \( u(z) = -\log \delta(z, \mathbb{C}^n \setminus \Omega) \) is continuous and \( \mathbb{P}_{SH}(\Omega) \).

**Proof.** Cont. of \( u \) follows from continuity of \( \delta(z, \mathbb{C}^n \setminus \Omega) \). The latter is \( \mathbb{K} \).

Pick \( z \in \Omega \), \( w \in \mathbb{C}^n \), \( t \in \mathbb{R} \), and consider \( V(t) = u(z + tw) \). Let \( K = \{ z + tw : \|z\| \leq 3 \} \subset \mathbb{C}^n \).

Clearly, by Max. Mod. Princ. in \( \mathbb{C} \), we conclude \( \delta(z + tw, \mathbb{C}^n \setminus \Omega) \leq K \). Let \( p(z) \) be any holom. polynomial in \( \mathbb{C}^n \), and \( Q(z) \) a holom. poly. in \( z \in \mathbb{C}^n \) s.t. \( p(z) = Q(z+w) \).

If \( V(t) \leq \Re(p(z)) \) on \( |t| = 3 \), then \( e^{V(t)} \leq |e^{p(z)}| \) on \( |t| = 3 \) or, equivalently,

\[ \delta(z + tw, \mathbb{C}^n \setminus \Omega)^{-1} \leq |e^{Q(z+tw)}|, \quad |t| = 3 \quad (1) \]

If we set \( p(z) = e^{-Q(z)} \in \mathcal{O}(\mathbb{C}^n) \), then inverting (1) \( \Rightarrow \)

\[ |f(z)| \leq \delta(z, \mathbb{C}^n \setminus \Omega), \quad z \in K. \]

By Prop 1, \( |f(z)| \leq \delta(z, \mathbb{C}^n \setminus \Omega) \), \( z \in K \), which in particular implies

\[ V(t) \leq \Re(p(z)), \quad |t| \leq 3. \]

This \( \Rightarrow \) \( V \) is \( \mathbb{K} \) in \( |t| < 3 \) \( \Rightarrow \) \( u \in \mathbb{P}_{SH}(\Omega) \).
Def. Let \( \Omega \subseteq \mathbb{C} \) and \( K \subseteq \mathbb{C}^n \).\n\[
K_\Omega^P = \{ z \in \Omega : u(z) \leq \sup_{z' \in K} u(z') , \forall u \in \text{PSH}(\Omega) \}.
\]

Rem. Note that \( f(\Omega) \Rightarrow u = \log |f| \in \text{PSH}(\Omega) \). Thus, if \( z \in K_\Omega^P \), then \( \log |f(z)| \leq \sup_{K} \log |f| \Rightarrow |f(z)| \leq \sup_{K} |f| \Rightarrow z \in K_\Omega^P \).
\[
\Rightarrow \ \overline{K_\Omega^P} \subseteq \overline{K_\Omega^P}.
\]

W/ \( \delta(z, c_n \Omega) \) as above (conv. + \( \delta(z) = 1 + \delta(z) \), etc.):

Thm 6. Let \( \Omega \subseteq \mathbb{C}^n \). TFAE:

(i) \( u(z) = -\log \delta(z, c_n \Omega) \in \text{PSH}(\Omega) \) or \( \Omega = c_n \).
(ii) \( \exists \text{ cont. } \text{PSH}(\Omega) \) from \( v(z) \leq 1 \). \( \overline{\Omega}_c := \{ z \in \Omega : v(z) \leq c \} \subseteq \Omega \), \( \forall c \in \mathbb{R} \).
(iii) \( \forall K \subseteq \mathbb{C}^n \), \( \overline{K} \subseteq \mathbb{C}^n \).

Rem. A consequence is that (i) holds for all \( \delta \) if it holds for some \( \delta \).

Pr. (i) \( \Rightarrow \) (iii). Take \( v(z) = 1 - \frac{1}{2} \log \delta(z, c_n \Omega) \) if \( \Omega \neq c_n \) and \( 1/2 \) if \( \Omega = c_n \).

(i) \( \Rightarrow \) (iii). Let \( K \subseteq \mathbb{C}^n \) and \( c = \sup_{K} v \). If \( z \in \overline{K} \), then \( v(z) \leq c \Rightarrow \overline{K} \subseteq \{ z \in \Omega : v(z) \leq c \} \).

Pr. (iii) \( \Rightarrow \) (i). Use the following lemma:

Lemma 1. If \( f : \Omega \rightarrow \Omega' \) is holomorph map and \( u \in \text{PSH}(\Omega') \), then \( u \circ f \in \text{PSH}(\Omega) \).

Pr. If \( u \in \mathbb{C}^2 \), then (iii) follows from Chain rule, since \( v = u \circ f \) satisfies
\[
\sum_{(i,j)} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j = \sum_{(i)} \left( \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \frac{P_{i,j}}{z_j} \right) w_i \bar{w}_j = \sum_{(i)} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \sum_{(i)} \left( P_{i,j} w_i \bar{w}_j \right)
\]
\[
= \sum_{(i)} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j \ w_i = \sum_{(i)} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j \Rightarrow \sum_{(i)} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j \geq 0.
\]

If \( u \in \text{PSH} \) but not \( \mathbb{C}^2 \), let \( u_\varepsilon \in \mathbb{C}^2 \cap \text{PSH} \) be regularization, \( \varepsilon > 0 \).

Since \( u_\varepsilon \triangleq u \), \( v_\varepsilon = u_\varepsilon \circ f \). By above \( v_\varepsilon \in \text{PSH} \). Now,

conclusion follows from Ex. If \( v_\varepsilon \neq v \), \( v_\varepsilon \in \text{PSH} \), \( \Rightarrow v \in \text{PSH} \).

(iii) \( \Rightarrow \) (i) Prf. \( \varepsilon > 0 \), \( \forall c \in \mathbb{C}^n \) and let \( \lambda \in \mathbb{C}, \lambda \leq \varepsilon, s.t.

Conclusion follows from $\frac{1}{2} x + t \leq \varepsilon$. 

(iii)$\Rightarrow$(i). Pick $z \in \Omega$, we $C^1$, and let $z \in \Omega$, $|t| \leq \varepsilon$, s.t.

$$D = \{z + tw : |t| \leq \varepsilon\} \subset \subset \Omega. \text{ Pick holom. poly. } f(t),$$

$$f(0) = 0, f(z) \text{ some holom. poly } \Omega(z), \text{ s.t.}$$

$$-\log \delta(z + tw, \Omega(z)) \leq \text{Re } f(t), \text{ } |t| = \varepsilon. \Rightarrow$$

$$\delta(z + tw, \Omega(z)) \geq e^{-f(t)}, \text{ } |t| = \varepsilon \text{ } (1)$$

Pick $a \in \Omega$, $\delta(a) < 1$, and consider $f(t) = z + tw + \lambda e^{-f(t)}$, $|t| \leq \varepsilon$, and $0 \leq \lambda \leq 1$. Let $D_a = \{f(t) : |t| \leq \varepsilon\}$. Have $D_0 = \overline{D} \subset \subset \Omega$.

Let $\Lambda = \{0 \leq \lambda \leq 1 : D_a \subset \subset \Omega\}$. We have $0 \in \Lambda$. Clearly, $\Lambda$ is open.

Claim: $\Lambda$ is closed.

Consider $K = \{z + tw + \lambda e^{-f(t)} : |t| = \varepsilon, 0 \leq \lambda \leq 1\}$. By (1), $K \subset \subset \Omega$.

Hence, $\forall \lambda \in \Lambda$, then $v(t) = u(z + tw + \lambda e^{f(t)})$ is SH on $D_\lambda$ by Lemma 1.

$$\Rightarrow \text{ for } |t| < \varepsilon, \text{ by HMP, } v(t) \leq \sup_{|t| < \varepsilon} v(t) \leq \sup_{K} v(t) \Rightarrow$$

$$z + tw + \lambda e^{-f(t)} \in \Omega \Rightarrow \overline{D}_a \subset \subset \Omega, \forall \lambda \in \Lambda.$$

By continuity, if $\lambda_0 \in \Lambda$, $\lambda_0 \rightarrow \lambda \Rightarrow \overline{D}_a \subset \subset \Omega, \forall \lambda \in \Lambda. \Rightarrow \text{ claim.}$

Thus, $\Lambda \neq \emptyset$, $\Lambda$ open, closed in $[0, 1] \Rightarrow \Lambda = [0, 1] \Rightarrow \overline{D} \subset \subset \Omega$, i.e. for any $|t| \leq \varepsilon$, $z + tw + \lambda e^{-f(t)} \in \Omega$. Moreover, since $a \in \Omega$ is arbitrary as long as $\delta(a) < 1$, we may conclude that the "$\delta$-ball" $\{z + tw + \lambda e^{f(t)} : a \in \Omega, \delta(a) < 1\}$ centered at $z + tw$ is contained in $\Omega$; i.e.

$$\delta(z + tw, \Omega(z)) \geq |e^{-f(t)}|, \forall |t| \leq \varepsilon.$$
centered at \( z^*+I_2W \) is contained in \( \Omega \); i.e.,
\[
\delta(z^*+I_2W, e^{u(x)}) \geq \lvert e^{f(z)} \rvert, \quad \forall \ l \leq 2.
\]

\[
\Leftrightarrow -\log \delta(z^*+I_2W, e^{u(x)}) \leq \text{Re } f(z), \quad 1 \leq \delta \Rightarrow -\log \delta(z, e^{u(x)}) \in \mathcal{P}(\Omega).
\]

Def. \( \Omega \subset \mathbb{C}^n \) is pseudoconvex if either (thus, all) conditions (i)-(iii) in Thm 6 holds.

Rem. The proof of (iii) \( \Rightarrow \) (i) in Thm 6 is an example of the Continuity Principle. Let \( \Omega \subset \mathbb{C}^n \) be \( \psi \text{cvx} \). Let \( \ell_x: \overline{D} \to \mathbb{C}^n \) a cont. family \( \{\ell_x\}_{x \in [0,1]} \) of holom. disks, \( (\pi \in \overline{D}) \cap (\mathbb{C}^n), \ell_x(\pi) \in \mathbb{C}(\mathbb{D} \times [0,1]) \)

s.t.
\[
\begin{align*}
\text{(1)} & \quad \ell_0(\overline{D}) \subset \Omega^c \\
\text{(2)} & \quad \ell_x(\partial D) \subset \Omega^c, \quad x \in [0,1]
\end{align*}
\]

Then, \( \{\ell_x(\overline{D}); x \in [0,1]\} \subset \Omega^c \).

Pf. Essentially done in (iii) \( \Rightarrow \) (i) above. \( \Box \)