Lecture 4

Recall: 

Cauchy's Formula in $D^n$: let $u$ cont. in $\overline{D^n}$, sep. hol. in $D^n$. Then

$$u(z) = \frac{1}{(2\pi i)^n} \oint_{\partial D^n} \frac{u(w)}{(w-z_1) \cdots (w-z_n)} dw_1 \cdots dw_n.$$ 

Consequences.

1) Theoretical sidetrack:

Hartogs' Theorem - I. Let $u$ be function $\mathbb{C} \to \mathbb{C}$ (no regularity assumed).

If $u$ is separately holomorphic in $\mathbb{C}$, then $u$ is holomorphic.

Remark: If $u \in C^1$, then this is essentially the def. of holomorphic. If $u$ is cont., then we proved this as cor. of Cauchy's Formula.

To prove without any reg. is more difficult and somewhat technical. It's an induction over the dimension and uses the power series expansion (below). We omit the pf and refer to Hormander.

2) Maximum Principle: If $u$ is holom. in $\Omega$, $|u|$ cannot achieve max. inside.

In fact:

Ex. Let $u$ be holom. in $D^n$, cont. on $\overline{D^n}$. Then

$$\max_{D^n} |u| = \max_{\partial D^n} |u|.$$ 

Power Series Expansion.

Def. A series $\sum a_k(z)$ converges normally in $\Omega \subseteq \mathbb{C}^n$ if

$$\sum_{a_k \in \mathbb{C}} \sup_{K \subseteq \Omega} |a_k(z)|$$

converges, if compact $K \subseteq \Omega$. 

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\[ \sum_{\alpha} \sup_{K} |a_{\alpha}(z)| \text{ converges, if compact } K \subset \Omega. \]

**Rem.** Normal conv. \( \Rightarrow \sum_{\alpha} a_{\alpha}(z) \) converges to function \( a(z) \). If \( a_{\alpha} \) are cont., then \( a \) is cont. If \( a_{\alpha} \) are holo, then \( a \) is holo.

**Thm.** If \( u \) is holo. in polydisk \( D^{m} \subset \mathbb{C}^{n} \) centered at \( a \in \mathbb{C}^{m} \), then
\[ u(z) = \sum_{\alpha \in \mathbb{Z}^{n}} u_{\alpha}(z-a)^{\alpha} \quad ; \quad u_{\alpha} = \frac{1}{\alpha!} \frac{\partial^{\alpha} u(a)}{\partial z^{\alpha}} \]

with normal convergence in \( D^{n} \).

**Note:** Here, \((z-a)^{\alpha} = (z_{1}-a_{1})^{\alpha_{1}} \ldots (z_{n}-a_{n})^{\alpha_{n}} \) and \( \alpha! = \alpha_{1}! \ldots \alpha_{n}! \).

**Proof:** Follows from Cauchy's Formula as in 1 variable. \( \Box \)

**Cauchy's Estimates.** Let \( D^{m} \subset \mathbb{C}^{n} \) be polydisk of polyradius \( r \), centered at \( a \). If \( u \) is holo. in \( D^{m} \), and \( |u| \leq M \) in \( D^{n} \), then
\[ |u_{\alpha}(a)| \leq \frac{M\alpha!}{r^{\alpha}}. \]

**Proof:** Follows from CF as in 1 variable. \( \Box \)

Another consequence of the power series expansion (as in \( \mathbb{C} \)) is **unique continuation**: If \( u \) is holom. \( u_{\alpha}(a) = 0 \ \forall \alpha \in \mathbb{Z}^{n} \), then \( u \equiv 0 \) in an open poly disk \( D^{m} \) centered at \( a \). If \( u \) holom. in \( \Omega \subset \mathbb{C}^{m} \), \( \Omega \) connected, and \( u \equiv 0 \) on open poly disk \( D^{m} \subset \Omega \), then \( u \equiv 0 \) in \( \Omega \).
Let $f$ be $(0,1)$-form and consider equation for for $u$:

$$\bar{\partial} u = f \quad (= \sum_{j=1}^{n} \bar{\partial}_j f_j \, dz_j).$$

Recall that necessary condition for solution is $\bar{\partial} f = 0$. As system:

$$\bar{\partial}_j u = f_j \quad \text{where} \quad \bar{\partial}_k \bar{\partial}_k \bar{\partial}_j u = \bar{\partial}_k \bar{\partial}_j u.$$

Recall: Given a form $f$ in $\Omega \subset \mathbb{C}^n$, the support of $f$ is

$$\text{supp } f = \{z \in \Omega : f(z) \neq 0\}.$$

Thus, let $f$ be $(0,1)$-form in $\Omega$ of class $C^k$ w/ compact support ($f \in C^k_0(\mathbb{C}^n)$). If $\bar{\partial} f = 0$, then $\exists u \in C^k_0(\mathbb{C}^n)$ s.t. $\bar{\partial} u = f$.

For $pf$, we first consider the case $n=1$:

Prop1. Let $f = \varphi \, dz$, $\varphi \in C^k_0(\mathbb{C})$. Then,

$$u(z) = \frac{1}{2\pi i} \int_{\text{C}} \frac{\varphi(z') \, d\bar{z}}{z' - z}$$

is $C^k$ in $\mathbb{C}$ and solves $\bar{\partial} u = f$.

Proof of Prop1. We recall that $\frac{1}{|z|}$ is integrable over any bdd open set $\Omega \subset \mathbb{C}$. Next, by (Ov),

$$u(z) = \bar{\partial}_{\bar{z}} \left( \frac{\varphi(z') \, dz'}{z' - z} \right) = \frac{1}{2\pi i} \int_{\text{C}} \frac{\varphi(z' + z) \, dz'}{z' - z} \quad (1)$$

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\[ U(z) = \{ \frac{z - z'}{d_{z} - d_{z'}} \} = \frac{1}{2\pi i} \int_{C} \frac{Q(z + z')dz'}{3'}, \quad (1) \]

so we see that \( u \) is as regular as \( Q \), i.e., \( C_k \)

in this case.

Now, let \( \Omega \) be an open set s.t. \( \text{supp } Q \subset \subset \Omega \).

Then, by General CF:

\[
Q(z) = \frac{1}{2\pi i} \left( \int_{z_0} \frac{Q(z')}{3 - z} dz' + \frac{Q(z)}{3 - z} \right)
\]

\[
= \frac{1}{2\pi i} \int_{\Omega} \frac{Q(z')}{3 - z} dz' = \frac{1}{2\pi i} \int_{\Omega} \frac{Q(z)(z + z')}{3'} dz' \quad (2)
\]

Thus, applying \( \partial_{\bar{z}} \) to (1) shows \( \partial_{\bar{z}} u = f \).

**Proof of Thm.** Consider

\[
U(z) = \frac{1}{2\pi i} \int_{C} \frac{f_1(3, z_2, \ldots, z_n)}{3 - z_1} dz_3 = \frac{1}{2\pi i} \int_{C} \frac{f_1(3 + z_1, z_2, \ldots)}{3'} dz_3
\]

As above, \( u \in C^k \) and since \( n > 1 \), \( u = 0 \) when \( |z_2| + |z_3| + \ldots + |z_n| \)

large.

\[ (z_2, \ldots, z_n) \quad \text{supp } u \quad \text{obtained by integrating over slice.} \]

\[ \xrightarrow{\text{supp } u} \]

By Prop.1, we have \( \partial_{\bar{z}_k} u = f_1 \). What about \( \partial_{\bar{z}_k} u \)?
By Prop 1, we have $\bar{\partial}_z u = f_1$. What about $\bar{\partial}_{\bar{z}} u$?

$$\bar{\partial}_{\bar{z}} u(z) = \frac{1}{2\pi i} \oint_C \frac{\bar{\partial}_{\bar{z}} u_1(z, z_2, \ldots, z_n)}{z} \, dz = \left\{ \begin{array}{l} \bar{\partial}_{\bar{z}} f_1 = \bar{\partial}_{\bar{z}} f_0 \end{array} \right.$$  

$$\begin{align*}
&= \frac{1}{2\pi i} \oint_C \frac{\bar{\partial}_{\bar{z}} f_0(z, z_2, \ldots, z_n)}{z} \, dz \\
&= \frac{1}{2\pi i} \oint_C \frac{\bar{\partial}_{\bar{z}} f_0(z, z_2, \ldots, z_n)}{z} \, dz = \text{Prop 1} \\
&= \frac{1}{2\pi i} \oint_C \frac{\bar{\partial}_{\bar{z}} f_0(z, z_2, \ldots, z_n)}{z - z_1} \, dz = f_0
\end{align*}$$

Thus, $u$ solves $\bar{\partial} u = f$. In particular, $\bar{\partial} u = 0$ outside supp $f \subset C$. Since $u \equiv 0$ when $|z_2|^2 + \cdots + |z_n|^2 \gg 1$, we conclude, by unique continuation, that $u \equiv 0$ outside some ball of radius $\gg 1$. Thus, $u$ has compact support. 

\begin{center}
\begin{tikzpicture}
  \draw (0,0) circle (1cm);
  \draw (0,1.5) -- (0.5,2);
  \draw (0,1.5) -- (-0.5,2);
  \draw (0,1.5) -- (0,-2);
  \draw (0,1.5) -- (-0.5,-2);
  \draw (0,1.5) -- (0.5,-2);
  \node at (0,3) {C line to integrate over};
  \node at (0,-3) {large ball $r$};
  \node at (0,4) {supp $f$};
  \node at (1.5,0) {$z_1$};
  \node at (0,2.5) {$(z_2, \ldots, z_n)$};
\end{tikzpicture}
\end{center}

Remark. The supp. of $u$ will be contained in the caison of $K$ and the odd components of $C^\infty K$. Thus, $u \equiv 0$ in the unOdd component of $C^\infty K$. 

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