Recall. \( \text{Thm 1}: \) \( \overline{\partial} \)-equation w/ compact support.

* Finish pf of Thm 1.

Haorgan Thm-II. \( \Omega \subseteq \mathbb{C}^n \) domain, \( K \subset \Omega \) compact such that \( \Omega \setminus K \) is connected. If \( u \) is holomorphic in \( \Omega \setminus K \), then \( u \) extends to a holom. fn. in \( \Omega \). (\( \forall \Omega \) holom. in \( \Omega \) s.t. \( \Omega \setminus K = u \).)

Remark. Spectacularly false in \( \mathbb{C} \). \( u(z) = \frac{1}{z} \) is holom. in \( \mathbb{D} \setminus \{0\} \), but \( u \) does not extend to \( \mathbb{D} \).

Prf: \( \text{let } \varphi \in C_c^\infty(\Omega) \text{ s.t. } \varphi = 1 \text{ on an open nbhd of } K. \)

Consider \( u_0 \in C_c^\infty(\Omega) \) given by

\[
\begin{cases}
(1-\varphi(z))u(z), & z \in \Omega \setminus K \\
0, & z \in K
\end{cases}
\]

Thus, \( u_0 = u \) on \( \Omega \setminus \text{supp } \varphi \).

Look for \( \mathcal{U} = u_0 + \mathcal{U} \) s.t. \( \mathcal{U} \) holom. in \( \Omega \), \( \mathcal{U} = u_0 \) in some open subset of \( \Omega \setminus \text{supp } \varphi \). Since \( \Omega \setminus \text{supp } \varphi \subseteq \Omega \setminus K \)
\( \Omega \setminus K \) is connected, this will imply \( \mathcal{U} = u \) on \( \Omega \setminus K \) as desired.

Now, \( \mathcal{U} \) holom. \( \Leftrightarrow \overline{\partial} \mathcal{U} = 0 \Leftrightarrow \overline{\partial}v = -\overline{\partial}u_0 \) s.t. \( f = -\overline{\partial}u_0 \).

Note that \( \overline{\partial}f = -\overline{\partial}^2 u_0 = 0 \). Moreover, since \( \text{supp } f = \text{supp } \varphi \setminus K \subset \Omega \),
Note that \( \overline{\Omega} = -\Omega \), Moreover, since \( \text{supp} f = \text{supp} f \cap \Omega \), we may extend \( f \) to \( C_0^\infty(\mathbb{C}^n) \) by defining it to be 0 in \( \mathbb{C}^n \setminus \Omega \). Clearly this preserves \( \overline{\Omega} = \Omega \). Now, by Thm 1, we can solve \( \overline{\Omega} v = f \) in \( \mathbb{C}^n \), and \( v \in C_0^\infty(\mathbb{C}^n) \) w/ \( v \equiv 0 \) in unbounded component \( \overline{\Omega} \setminus \text{supp} f \). With this choice of \( v \), \( U \) is holomorphic in \( \Omega \) and \( U = u_0 \) in \( \overline{\Omega} \cap \Omega \), which intersects \( \Omega \setminus \overline{K} \). This completes the pf. \( \Box \)

**Prop 1.** Let \( u \) be holomorphic in a polydisk \( D^* \subseteq \mathbb{C}^n \) and let \( Z_u = \{ z \in D^* : u(z) = 0 \} \).

If \( Z_u \neq \emptyset \), then \( Z_u \) is not compact in \( D^* \).

**Rem.** Clearly not true in a disk \( D \subseteq \mathbb{C}^n \).

**Pf.** Suppose \( Z_u \neq \emptyset \), \( Z_u \subseteq D^* \) compact. Then \( \exists \) smaller polydisk \( E^* = E_1 \times \ldots \times E_n \), \( \overline{E_j} \subseteq D_j \), such that \( Z_u \subseteq E^* \). But \( \overline{E^*} \) is compact in \( D^* \), \( D^* \setminus \overline{E^*} \) connected, and \( v = \frac{1}{u} \) is holomorphic in \( D^* \setminus \overline{E^*} \). By Hartogs -II, \( v \) is holomorphic in \( D^* \), i.e. across \( Z_u \), which is easily seen to be impossible. \( \Box \)

**Boundary version of Hartogs -II.**

**Thm 2.** Let \( \Omega \subseteq \mathbb{C}^n \) be bdd, \( \mathbb{C}^n \setminus \overline{\Omega} \) connected, and \( \partial \Omega \) smooth, i.e. \( \exists c \in \mathbb{C}^n \) s.t. \( \partial \Omega = \{ z \in \mathbb{C}^n : \partial_{\overline{\Omega}} \} \). If \( u \in C_0^\infty(\Omega) \) and \( \overline{\partial u} = 0 \) on \( \partial \Omega \) (a defining beam for \( \partial \Omega \)), then \( \exists \nu \in C_0^\infty(\overline{\Omega}) \)
$\overline{\partial} u = 0 \text{ on } \mathbb{R}^2 \ (u|_{\mathbb{R}^2} \text{ is CR}), \text{ then } \exists v \in \mathcal{C}^{\omega}(\mathbb{R}^2)$

s.t. $v$ holomorphic in $\mathbb{R}^2$ and $v|_{\mathbb{R}^2} = u$.

Rem. 1. If such $v$ exists, $v-u = \alpha \beta \implies \overline{\partial}v - \overline{\partial}u = \alpha \partial \overline{\beta}$ on $\mathbb{R}^2 = \{\beta = 0\}$. Thus, since $v$ holomorphic, $\overline{\partial}v = \alpha \partial \overline{\beta} = 0$.

2. Ex. $\overline{\partial} u \overline{\beta} u$ can be reformulated as

(2) $\overline{\partial} u \overline{\beta} u : \sum_{j=1}^n c_j \frac{\partial u_j}{\partial \overline{z}_j} = 0, \forall C = (c_1, \ldots, c_n) : \sum_{j=1}^n \frac{\partial u_j}{\partial \overline{z}_j} c_j = 0$

This is a condition that depends only on $u|_{\mathbb{R}^2}$. Functions $u \in \mathcal{C}^{\omega}(\mathbb{R}^2)$ that satisfy (2) are called CR functions.