Lecture 12-14

Wednesday, February 12, 2020 5:30 AM

Review of subharmonic fcns in $\Omega \subseteq \mathbb{C}$:

Def 1. A function $u: \Omega \subseteq \mathbb{C} \to [-\infty, \infty)$ is subharmonic (SH)

(i) u is upper semicont. (usc); i.e. $\{z \in \Omega : u(z) < c\}$ is open $\forall c \in \mathbb{R}$. (not $\equiv -\infty$).

(ii) $\forall K \subset\subset \Omega$, $h \in C(K)$ and h harmonic in $\mathring{K}$, if $u \le h$ on ∂K, then $u \le v$ in K. (locally; $\forall a \in \Omega$, $\mathbb{D}_\varepsilon = \{|z-a| < \varepsilon\}$, $K \subset\subset \mathbb{D}_\varepsilon$.)

Thm 1. Let $u: \Omega \to [-\infty, \infty)$ be usc. TFAE:

(i) u is SH.

(ii) $\forall \bar{D} \subset\subset \Omega$ closed disk, f holom. polynomial, if $u \le \operatorname{Re} f$ in ∂D, then $u \le \operatorname{Re} f$ in D.

(iii) $\forall \bar{D} = \{z \in \Omega : |z-a| \le r\} \subset\subset \Omega$ closed disk, probability measure μ on $[0, r]$, it holds that

$$u(a) \le \frac{1}{2\pi} \int_0^{2\pi} \int_0^r u(a + te^{i\theta})\, d\theta\, d\mu(t)$$

Rem: • (iii) is called submeanvalue property, usually used with $\mu = \delta_r$, delta mass at $t = r$.

• u is harmonic $\iff$ $u, -u$ are SH.

Pf. Hörmander.

← Lecture 13 starts here

Prop 1. If $u \in C^2$, then $u \in SH \iff \Delta u \ge 0$.

Pf. Hörmander or see below. ∎

Recall: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$.

Rem. One first needs to prove that $u \in SH\ (PSH) \Rightarrow u \in L^1_{loc}$, integrable over compact sets. We refer to Hörmander.

Thm 2. Let $u \in SH(\Omega)$, $\Omega \subseteq \mathbb{C}$. Then, we have

$$\int_\Omega u \Delta \varphi \, d\lambda \ge 0, \quad \forall \varphi \in C^2_0(\Omega),\ \varphi \ge 0.$$

↑ Lebesgue measure

Rem. This yields "$\Rightarrow$" in Prop 1 by integration by parts and φ approximating δ_z, $z \in \Omega$.

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Pf. Pick $\varphi \in \mathcal{C}_0^2(\Omega)$, $0 < r < d(\operatorname{supp}\varphi, \mathbb{C} \setminus \Omega)$. Then, for $z \in \operatorname{supp}\varphi$

$$u(z) \le \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\,d\theta.$$

$$\Rightarrow \int_\Omega u(z)\varphi(z)\,d\lambda_z \le \frac{1}{2\pi}\int_\Omega\int_0^{2\pi} \varphi(z)\,u(z+re^{i\theta})\,d\theta\,d\lambda = \left\{ \begin{matrix} z' = z + re^{i\theta} \\ d\lambda' = d\lambda \end{matrix} \;\middle|\; \operatorname{supp}\varphi \subset\subset \Omega \right\}$$

$$= \frac{1}{2\pi}\int_\Omega\int_0^{2\pi} \varphi(z - re^{i\theta})\,u(z)\,d\theta\,d\lambda \Rightarrow$$

$$0 \le \frac{1}{2\pi}\int_\Omega u(z)\int_0^{2\pi}\left[\varphi(z - re^{i\theta}) - \varphi(z)\right]d\theta\,d\lambda \qquad (1)$$

Note $\varphi(z - re^{i\theta}) - \varphi(z) = \varphi_x(z)\,r\cos\theta + \varphi_y(z)\,r\sin\theta + \frac{1}{2}\big(\varphi_{xx}(z)\,r^2\cos^2\theta +$
$\varphi_{yy}(z)\,r^2\sin^2\theta + 2\varphi_{xy}(z)\,r^2\sin\theta\cos\theta\big) + o(r^2)$.

Since $\int_0^{2\pi}\cos\theta\,d\theta = \int_0^{2\pi}\sin\theta\,d\theta = \int_0^{2\pi}\cos\theta\sin\theta\,d\theta = 0$, (1) $\Rightarrow$

$$0 \le \frac{1}{2\pi}\int_\Omega u(z)\int_0^{2\pi}\left[\Delta\varphi(z)\frac{r^2}{2} + o(r^2)\right]d\theta\,d\lambda.$$

Now, multiply by $\frac{2}{r^2}$ and let $r \to 0 \Rightarrow \quad 0 \le \int_\Omega u(z)\,\Delta\varphi(z)\,d\lambda$ as desired. $\square$

Plurisubharmonic fcns in $\Omega \subseteq \mathbb{C}^n$.

Def. 2 A function $u: \Omega \to [-\infty, \infty)$ is plurisubharmonic (PSH) if

(i) u is usc. (and not $\equiv -\infty$)

(ii) $\forall z \in \Omega$, $w \in \mathbb{C}^n$, $\tau \in \mathbb{D}_\varepsilon = \{\tau \in \mathbb{C} : |\tau| < \varepsilon\}$, $\tau \to u(z + \tau w)$ is SH in $\mathbb{D}_\varepsilon$. (i.e. restriction to each cplx line is SH).

Ex. If $f \in \mathcal{O}(\Omega)$, then $u = \log|f|$ is PSH.

Thm 3. If $u \in \mathcal{C}^2(\Omega)$ then $u \in PSH \Leftrightarrow$

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$$\forall z \in \Omega, w \in \mathbb{C}^n: \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(z) w_i \bar{w}_j \geq 0. \qquad (1)$$

Pf. Pick $z \in \Omega$, $w \in \mathbb{C}^n$ and consider $v(\tau) = u(z + \tau w)$ in $\mathbb{D}_\varepsilon$. Must show that (1) $\Leftrightarrow$ v is SH. By Rem., v SH $\Leftrightarrow \Delta v = 4 \partial_\tau \partial_{\bar\tau} v \geq 0$. But, since $z \in \Omega$ arbitrary, suffices to check $(\partial_\tau \partial_{\bar\tau} v)(0) \geq 0$. Chain rule $\Rightarrow$ $\boxtimes$.

Regularization.

Thm 4. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ s.t. $\varphi \geq 0$, $\operatorname{supp}\varphi \subset\subset \{|z| \leq 1\}$, $\varphi(z) = \varphi(|z_1|, \dots, |z_n|)$, and $\int_{\mathbb{C}^n} \varphi \, d\lambda = 1$. If $u \in PSH(\Omega)$, then

$$u_\varepsilon(z) = \int_{\mathbb{C}^n} u(z - \varepsilon\zeta)\varphi(\zeta)\, d\lambda_\zeta$$

is $\mathcal{C}^\infty$ and PSH in $\Omega_\varepsilon = \{z \in \Omega, d(z, \mathbb{C}^n \setminus \Omega) > \varepsilon\}$ and $\underbrace{u_\varepsilon \searrow u}_{\text{pointwise decreasing}}$ as $\varepsilon \searrow 0$.

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We shall prove this for $n=1$. General n follows in a similar way.

Pf for $n=1$. Note that $1 = \int_{\mathbb{C}} \varphi(z)\, d\lambda = \int_0^{2\pi}\int_0^1 \varphi(\rho)\rho\, d\rho\, d\theta = 2\pi \int_0^1 \varphi(\rho)\rho\, d\rho =$

$$\left\{ \begin{array}{l} \rho = \frac{\rho'}{\varepsilon} \\ d\rho = \varepsilon^{-1} d\rho' \end{array} \middle| \begin{array}{l} 1 \to \varepsilon \\ 0 \to 0 \end{array} \right\} = 2\pi \int_0^\varepsilon \frac{1}{\varepsilon^2}\varphi\left(\frac{\rho'}{\varepsilon}\right)\rho' d\rho'.$$

Consider, for $\varepsilon > 0$, the prob. measure $\mu_\varepsilon(\rho) = 2\pi \frac{1}{\varepsilon^2}\varphi\left(\frac{\rho}{\varepsilon}\right)\rho\, d\rho$. By Thm 1 (iii), we have for $z \in \Omega_\varepsilon$

$$u(z) \leq \frac{1}{\cancel{2\pi}} \int_0^{2\pi}\int_0^\varepsilon u(z - \rho e^{i\theta}) \cancel{2\pi} \frac{1}{\varepsilon^2} \varphi\Big(\frac{\rho}{\varepsilon}\overset{e^{i\theta} \text{ since } \varphi = \varphi(|\zeta|)}{}\Big)\rho\, d\rho\, d\theta = \left\{ \begin{array}{l} \zeta = \frac{\rho}{\varepsilon} e^{i\theta} \\ d\lambda_\zeta = \frac{1}{\varepsilon^2}\rho\, d\rho\, d\theta \end{array} \right\}$$

$$= \int u(z - \varepsilon\zeta)\varphi(\zeta)\, d\lambda_\zeta \overset{\zeta' = z - \varepsilon\zeta}{=} \int u(\zeta) \frac{1}{\varepsilon^2}\varphi\left(\frac{z - \zeta}{\varepsilon}\right) d\lambda_\zeta \qquad (1)$$

$$= \underbrace{\int_\Omega u(z-\varepsilon\zeta)\,\varphi(\zeta)\,d\lambda_\zeta}_{u_\varepsilon(z)} \overset{\downarrow}{=} \int_\Omega u(\zeta)\,\frac{1}{\varepsilon^2}\,\varphi\!\left(\frac{z-\zeta}{\varepsilon}\right)d\lambda_\zeta \qquad (1)$$

Obviously $u(z) \le \liminf_{\varepsilon\to 0} u_\varepsilon(z)$. But usc of u also implies

$\limsup_{\varepsilon\to 0} u_\varepsilon(z) \le u(z)$ (<u>Ex.</u>) $\Rightarrow u(z) = \lim_{\varepsilon\to 0} u_\varepsilon(z)$.

It is clear from last identity in (1) that $u_\varepsilon \in C^\infty$. We note that

$$\Delta u_\varepsilon(z) = \int_\Omega u(\zeta)\,\Delta\!\left(\frac{1}{\varepsilon^2}\varphi\!\left(\frac{z-\zeta}{\varepsilon}\right)\right)d\lambda_\zeta \ge 0 \quad \text{by Thm 2.}$$

Thus, by Prop1, u_ε is SH. But let's actually prove it, since we will need this calculation to show that $u_\varepsilon(z) \searrow u(z)$ (remaining part of Thm 4).

<u>Claim</u>. If $u \in C^2$, $\Delta u \ge 0$, then $u \in SH$.

Consider $M(r) = \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\,d\theta$, $r>0$ small ($\overline{B(z,r)} \subset\subset \Omega$).

In polar coordinates $\zeta = re^{i\theta}$, $\Delta_\zeta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$. Note that

$\int_0^{2\pi} \frac{\partial^2}{\partial\theta^2}\left(u(z+re^{i\theta})\right)d\theta = 0$ by periodicity $\Rightarrow$

$$\underbrace{\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)M(r)}_{\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}M(r)\right)} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)u(z+re^{i\theta})}_{=\Delta_\zeta[u(z+\zeta)] = (\Delta u)(z+\zeta)}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}(\Delta u)(z+re^{i\theta})\,d\theta \ge 0.$$

$\Rightarrow rM'(r) \nearrow$ as $0<r\nearrow$. Since $rM'(r) \to 0$ as $r \to 0$, for $r>0$,

$M'(r) \ge 0 \Rightarrow M(r) \nearrow$ as $0<r\nearrow$. Since $M(r) \to u(z)$ (previous Ex.)

We conclude that $u(z) \le M(r) = \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})\,d\theta$, i.e. $u \in SH$.

This proves claim, but also that if <u>$u \in C^2$</u>, then $z \in \Omega_\varepsilon$

$$u_\varepsilon(z) = \int_\Omega u(z+\varepsilon\zeta)\,\varphi(\zeta)\,d\lambda = \int_0^1 \underbrace{\int_0^{2\pi} u(z+\varepsilon re^{i\theta})\,d\theta}\;\varphi(r)\,r\,dr$$

Ω $\qquad$ $0 \quad \delta$

is $\searrow$ as $\varepsilon \searrow 0$, since $2\pi M(r\varepsilon) \searrow$ as $\varepsilon \searrow$.

Now, we show that for general $u \in SH$, $u_\varepsilon(z) \searrow u(z)$ (already know $u_\varepsilon(z) \to u(z)$). Consider $u_{\varepsilon,\delta}(z) = \int_\Omega u_\delta(z+\varepsilon\zeta)\varphi(\zeta)\,d\lambda_\zeta$.

Since $u_\delta \to u$ pointwise (per above), and $u \le M$ on compacts $\Rightarrow$ $u \le u_\delta \le M$, so Lebesgue Dominated Convergence $\Rightarrow u_\delta \to u$ in L^1 over compact sets. $\Rightarrow \lim_{\delta\to 0} u_{\varepsilon,\delta}(z) = \int_\Omega u(z+\varepsilon\zeta)\varphi(\zeta)\,d\lambda_\zeta = u_\varepsilon(z)$

By the observation above, for fixed $\delta > 0$, $u_{\varepsilon,\delta}(z) \searrow u_\delta(z)$ as $\varepsilon \searrow 0$.
I.e. if $\varepsilon_1 < \varepsilon_2 \Rightarrow u_{\varepsilon_1,\delta}(z) \le u_{\varepsilon_2,\delta}(z)$. Taking limits $\delta \to 0 \Rightarrow$ $u_{\varepsilon_1}(z) \le u_{\varepsilon_2}(z) \Rightarrow u_\varepsilon(z) \searrow u(z)$ as desired. $\boxtimes$

Pseudoconvexity.

Recall. If $\delta(z) = \max(|z_1|, \ldots, |z_n|)$ and $\delta(z, \mathbb{C}^n \setminus \Omega) = \inf_{w \in \mathbb{C}^n\setminus\Omega} \delta(z-w)$, then:

Prop 1. If $\Omega \subseteq \mathbb{C}^n$ is d.o.holom., $K \subset\subset \Omega$, $f \in \mathcal{O}(\Omega)$ s.t. $|f(z)| \le \delta(z, \mathbb{C}^n\setminus\Omega)$ on K, then $|f(z)| \le \delta(z, \mathbb{C}^n\setminus\Omega)$ in $\hat{K}_\Omega$.

Rem. The conclusion can be proved w/ any continuous function $\delta(z)$ s.t. $\delta(tz) = |t|\delta(z)$, $t \in \mathbb{C}$. E.g. $\delta(z) = \left(\sum_{j=1}^n |z_j|^p\right)^{1/p}$, L^p-norm.

Thm 5. Let $\Omega \subseteq \mathbb{C}^n$ d.o.holom. and $\delta(z) > 0$, cont. s.t. $\delta(tz) = t\delta(z)$ for $t > 0$. Then $u(z) = -\log\delta(z, \mathbb{C}^n\setminus\Omega)$ is continuous and $PSH(\Omega)$.

Pf. Cont. of u follows from continuity of $\delta(z, \mathbb{C}^n\setminus\Omega)$. The latter is Ex..
Pick $z^0 \in \Omega$, $w \in \mathbb{C}^n$, $\tau \in \mathbb{C}$, $|\tau| \le \varepsilon$, and consider $v(\tau) = u(z^0 + \tau w)$. Let $K = \{z^0 + \tau w \in \Omega : |\tau| = \varepsilon\} \subset\subset \Omega$. Clearly, by Max. Mod. Princ. in τ-plane, we conclude $\{z^0 + \tau w \in \Omega : |\tau| \le \varepsilon\} \subseteq \hat{K}_\Omega$. Let $P(\tau)$ be any holom. polynomial in $\tau \in \mathbb{C}$, and $Q(z)$ a holom. poly.

$|\tau| = \varepsilon$

$p(\tau)$ be any holom. polynomial in $\tau \in \mathbb{C}$, and $Q(z)$ a holom. poly. in $z \in \mathbb{C}^n$ s.t. $p(\tau) = Q(z^0 + \tau w)$. If $v(\tau) \le \operatorname{Re} p(\tau)$ on $|\tau| = \varepsilon$, then $e^{v(\tau)} \le |e^{p(\tau)}|$ on $|\tau| = \varepsilon$ or, equivalently,

$$\delta(z^0 + \tau w, \mathbb{C}^n \setminus \Omega)^{-1} \le |e^{Q(z^0 + \tau w)}| \quad , \ |\tau| = \varepsilon \qquad (1)$$

If we set $f(z) = e^{-Q(z)} \in \mathcal{O}(\Omega)$, then inverting (1) $\Rightarrow$

$$|f(z)| \le \delta(z, \mathbb{C}^n \setminus \Omega) \quad , \quad z \in K.$$

By Prop 1, $|f(z)| \le \delta(z, \mathbb{C}^n \setminus \Omega)$, $z \in \hat{K}_\Omega$, which in particular implies

$$v(\tau) \le \operatorname{Re} p(\tau) \ , \ |\tau| \le \varepsilon.$$

This $\Rightarrow$ v is SH in $|\tau| < \varepsilon$ $\Rightarrow$ $u \in PSH(\Omega)$. $\boxtimes$

<u>Def</u>: Let $\Omega \subseteq \mathbb{C}^n$ and $K \subset\subset \Omega$. The <u>$PSH(\Omega)$-hull</u>, $\hat{K}^P_\Omega \subseteq \Omega$ is

$$\hat{K}^P_\Omega = \{ z \in \Omega : u(z) \le \sup_{\zeta \in K} u(\zeta), \ \forall u \in PSH(\Omega) \}.$$

<u>Rem.</u> Note that $f \in \mathcal{O}(\Omega) \Rightarrow u = \log|f| \in PSH(\Omega)$. Thus, if $z \in \hat{K}^P_\Omega$, then $\log|f(z)| \le \sup_K \log|f| \Rightarrow |f(z)| \le \sup_K |f| \Rightarrow z \in \hat{K}_\Omega$.

$\Rightarrow \boxed{\hat{K}^P_\Omega \subseteq \hat{K}_\Omega}$.

W/ $\delta(z, \mathbb{C}^n \setminus \Omega)$ as above (cont. + $\delta(tz) = |t|\delta(z)$, $t \in \mathbb{C}$):

<u>Thm 6</u>. Let $\Omega \subseteq \mathbb{C}^n$. TFAE:

(i) $u(z) = -\log \delta(z, \mathbb{C}^n \setminus \Omega)$ is $PSH(\Omega)$

(ii) $\exists$ cont. $PSH(\Omega)$ fcn $v(z)$ s.t. $\Omega_c := \{ z \in \Omega : u(z) < c \} \subset\subset \Omega$, $\forall c \in \mathbb{R}$.

(iii) $\forall K \subset\subset \Omega$, $\hat{K}^P_\Omega \subset\subset \Omega$.

<u>Rem.</u> A consequence is that (i) holds for all δ if it holds for some δ.

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