ON THE ANALYTICITY OF CR MAPPINGS BETWEEN NONMINIMAL HYPERSURFACES

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Abstract. Let $M \subset \mathbb{C}^2$ be a connected real-analytic hypersurface containing a connected complex hypersurface $E \subset \mathbb{C}^2$, and let $f\colon M \to \mathbb{C}^2$ be a smooth CR mapping sending M into another real-analytic hypersurface $M' \subset \mathbb{C}^2$. In this paper, we prove that if f does not collapse E to a point and does not collapse M into the image of E, and if the Levi form of M vanishes to first order along E, then f is real-analytic in a neighborhood of E. In general, the corresponding statement is false if the Levi form of M vanishes to second order or higher, in view of an example due to the author. We also show analogous results in higher dimensions provided that the target M' satisfies a certain nondegeneracy condition.

The main ingredient in the proof, which seems to be of independent interest, is the prolongation of the system defining a CR mapping sending M into M' to a Pfaffian system on M with singularities along E. The nature of the singularity is described by the order of vanishing of the Levi form along E.

1. Introduction

Let M be a real-analytic hypersurface in $\mathbb{C}^{n+1}$. As a submanifold of complex space, M inherits a partial complex structure—a *CR structure*—in the following way (which, of course, is well known; see e.g. the books [B91] and [BER99] for basic concepts and notions in CR geometry). The *CR bundle* $\mathcal{V}$ of M, yielding the structure, is a rank n subbundle of $\mathbb{C}TM$, the complexified tangent bundle on M, defined by

$$\mathcal{V} := T^{0,1}\mathbb{C}^{n+1} \cap \mathbb{C}TM,$$

where $T^{0,1}\mathbb{C}^{n+1}$ denotes the bundle of $(0,1)$ tangent vectors on $\mathbb{C}^{n+1}$. The two basic properties of $\mathcal{V}$ are the following: (a) The commutator of two sections of $\mathcal{V}$ is again a section of $\mathcal{V}$ (formal integrability); (b) $\mathcal{V} \cap \bar{\mathcal{V}}$ consists of only the zero section. Sections of the CR bundle will be called *CR vector fields*.

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Let $L_{\bar{1}},\dots,L_{\bar{n}}$ denote a basis for the CR vector fields on M near a point $p_0\in M$. A smooth mapping $f\colon M\to\mathbb{C}^k$, with $f=(f_1,\dots f_k)$, is called *CR* (near p_0) if

$$L_{\bar{A}}f_j=0,\quad 1\le A\le n,\ 1\le j\le k. \tag{1}$$

If f sends M into another real-analytic hypersurface $M'\subset\mathbb{C}^k$, i.e. if f satisfies a nonlinear equation

$$\rho'(f,\bar{f})=0, \tag{2}$$

for some real-analytic function $\rho'(Z,\bar{Z})$ with $d\rho'\neq 0$, then f is CR if and only if the tangent mapping $f_*\colon\mathbb{C}TM\to\mathbb{C}TM'$ sends the CR bundle $\mathcal{V}$ of M into the CR bundle $\mathcal{V}'$ of M'. In this paper, we shall address the problem of describing conditions which imply that a CR mapping f sending M into M' is necessarily real-analytic near p_0. We should point out that a CR mapping f is real-analytic near p_0 if and only if f extends holomorphically to a neighborhood of p_0 in $\mathbb{C}^{n+1}$ (see e.g. [BER99], Chapter I).

There is an extensive literature on this subject if M is assumed to be *minimal* at p_0, i.e. if there are no complex hypersurfaces through p_0 in $\mathbb{C}^{n+1}$ contained in M. In this case, all CR mappings extend holomorpically to one side of M (see [Tr86]; cf. also [BT84], [Tu88]) and results on the problem decribed above are often referred to as reflection principles. We mention here the papers [P75], [Le77], [BJT85], [BR88], [DF88], [DW80], [Me95], [Hu96], and refer the reader to the notes in [BER99], Chapter IX, and the survey article [Hu98] for a more detailed history. We should point out, however, that most results mentioned above require additional nondegeneracy conditions on the manifolds and mappings which do not appear to be necessary conditions. Thus, even though much is known, the problem is still not completely resolved in the minimal case.

In the present paper, we shall assume that M is not minimal at p_0; i.e. there exists a complex hypersurface $E\subset\mathbb{C}^{n+1}$ (which we shall assume is connected) through p_0 contained in M. To state our main results, we need to define an invariant which measures the order of vanishing of the Levi form along E. To be more precise, we must introduce some notation. It is well known ([CM74]; see also [BER99], Chapter IV]) that, for any real-analytic hypersurface $M\subset\mathbb{C}^{n+1}$ (minimal or not) and $p_0\in M$, there are local holomorphic coordinates $(z,w)\in\mathbb{C}^n\times\mathbb{C}$ vanishing at p_0 such that M is defined locally near $p_0=(0,0)$ by an equation of the form

$$\operatorname{Im} w=\phi(z,\bar{z},\operatorname{Re} w), \tag{3}$$

where $\phi(z,\chi,s)$ is a holomorphic function satisfying

$$\phi(z,0,s)=\phi(0,\chi,s)=0. \tag{4}$$

Such coordinates are called *normal coordinates* for M at p_0. Although they are not unique, one can define an integer $m\geq 0$, which is independent of the choice of normal coordinates, as follows. If $\phi(z,\bar{z},s)\equiv 0$, we set $m=\infty$; otherwise, we define m by Taylor expanding in s,

$$\phi(z,\chi,s)=\phi_m(z,\chi)s^m+O(s^{m+1}), \tag{5}$$

such that $\phi_m(z,\chi)\not\equiv 0$. The fact that the integer m is a biholomorphic invariant of M at p_0 was first observed in [Me95]. If $m=\infty$, i.e. if $\phi(z,\bar{z},s)\equiv 0$, then M is said to be *Levi flat*.

When M is not minimal at p_0, the complex hypersurface $E\subset M$ is given, in normal coordinates, by $w=0$ and the fact that M is not minimal at p_0 is therefore characterized by $m\geq 1$. It is not difficult to see that the integer m, which *a priori* depends on $p_0\in E\subset M$, is constant along E. Indeed, as mentioned above, m measures the order of vanishing of the Levi form of M along the complex hypersurface E (see Proposition 3.1 for the precise statement). We shall say that M is of *m-infinite type* along E (or at p_0); recall that for a real-analytic hypersurface M, being nonminimal at p_0 is equivalent to being of infinite type in the sense of Kohn [K72] and Bloom–Graham [BG77] at p_0. One of our main results, for $n=1$, is the following.

Theorem 1.1. *Let $M\subset\mathbb{C}^2$ be a real-analytic, connected hypersurface which is of 1-infinite type along a connected complex curve (or, equivalently, hypersurface) $E\subset\mathbb{C}^2$ contained in M. Let $f\colon M\to\mathbb{C}^2$ be a C^∞-smooth CR mapping. Assume that $f(M)$ is contained in a real-analytic hypersurface $M'\subset\mathbb{C}^2$, and that*

- (i) *$f(M)\not\subset f(E)$;*
- (ii) *$f|_E$ is not constant.*

Then, f extends as a holomorphic mapping from an open neighborhood of E in $\mathbb{C}^2$, i.e. there exists a holomorphic mapping $F\colon U\to\mathbb{C}^2$, where U is an open neighborhood of E in $\mathbb{C}^2$, such that $F|_M=f$ (in $M\cap U$).

Theorem 1.1 will follow from a more general, but slightly more technical result which is valid in any number of dimensions. However, before explaining this more general result we make a few remarks. First, we would like to point out that Theorem 1.1 fails in general if M is of m-infinite type along E with $m\geq 2$ as the following example from [E96] shows.

Example 1.2. Let $M\subset\mathbb{C}^2$ be defined by

$$\operatorname{Im} w=\theta(\arctan|z|^2,\operatorname{Re} w), \tag{6}$$

where $t=\theta(\xi,s)$ is the unique solution of $\xi(s^2+t^2)-t=0$ with $\theta(0,0)=0$. One can show that M is real-analytic near $(z,w)=(0,0)$ and of 2-infinite type along the complex curve $E:=\{w=0\}$ which is contained in M. The restriction f to M of the mapping $F(z,w):=(z,g(w))$, where $g(w)=e^{-1/w}$ if $\operatorname{Re} w>0$ and $g(w)=0$ if $\operatorname{Re} w\leq 0$, is a C^∞-smooth CR mapping which does not extend holomorphically to any neighborhood of $(0,0)$. (Indeed, the second component of

f vanishes to infinite order along E.) Moreover, it is shown in [E96] that $f(M)$ is contained in the real-analytic (indeed, real-algebraic) hypersurface $M' \subset \mathbb{C}^2$ defined by

$$\text{Im } w = (\text{Re } w)|z|^2. \tag{7}$$

Observe that conditions (i) and (ii) of Theorem 1.1 are both satisfied.

The CR mapping in Example 1.2 does not extend to either side of the hypersurface M near E. Indeed, the conclusion of Theorem 1.1, with M of m-infinite type for any $1 \leq m < \infty$ and f merely assumed to be continuous, was proved in [EH00] under the *additional assumption* that the CR mapping extends holomorphically to one side of M. As shown by Example 1.2, the conclusion fails for $m \geq 2$ without assuming one-sided extension. It is also noteworthy that, without this assumption, the conclusion fails with C^∞-regularity replaced by C^k-regularity for any finite k (in contrast, again, with the result in [EH00]) as is shown by the following example from [EH00].

Example 1.3. Let $M \subset \mathbb{C}^2$ be defined by

$$\text{Im } w = (\text{Re } w)|z|^2, \tag{8}$$

and $M'_k \subset \mathbb{C}^2$ by

$$\text{Im } w = (\text{Re } w)h_k(z, \bar{z}), \tag{9}$$

where h_k is defined as follows. Let

$$\begin{aligned} \phi_k(s,t) &:= \text{Re } ((s+it)^k) = s^k + \sum_{j=0}^{k-1} a_j s^j t^{k-j} \\ \psi_k(s,t) &:= \text{Im } ((s+it)^k) = \sum_{j=0}^{k-1} b_j s^j t^{k-j} \end{aligned} \tag{10}$$

and define

$$h_k(z,\bar{z}) := \frac{\sum_{j=0}^{k-1} b_j |z|^{2(k-j)}}{1 + \sum_{j=0}^{k-1} a_j |z|^{2(k-j)}}, \tag{11}$$

where the a_j and b_j are defined by (10). The mapping $F(z,w) := (z, g(w))$, with $g(w) = -w^k$ for Re $w \leq 0$ and $g(w) = w^k$ for Re $w > 0$ is a C^{k-1}-differentiable CR mapping from M into M'_k, which satisfies (i) and (ii) of Theorem 1.1 but does not extend holomorphically to a neighborhood of 0 in $\mathbb{C}^2$. Observe that M is of 1-infinite type along $w = 0$.

In order to state the more general result for real-analytic hypersurfaces in higher dimensional complex space, we need the following notion from [Me95]. Assume that M is not Levi flat and write the defining equation (3) of M near $p_0 = (0,0)$ as

$$\text{Im } w = (\text{Re } w)^m \psi(z, \bar{z}, \text{Re } w), \tag{12}$$

where m is the integer defined by (5), and also write

$$\psi(z, \chi, 0) = \sum_{\alpha} a_\alpha(z) \chi^\alpha. \tag{13}$$

Then, we shall say that M is *m-essential* at $p_0 = (0,0)$ if the ideal generated by the collection $a_\alpha(z)$, $\alpha \in \mathbb{Z}^n_+$, is of finite codimension in the ring $\mathbb{C}\{z\}$ of convergent power series. This notion is analogous to that of essential finiteness in the finite type case and was shown, in [Me95], to be independent of the choice of normal coordinates for M at p_0. If M is of m-infinite type at p_0, for some $1 \leq m < \infty$, and M is m-essential at p_0, then we shall also say that M is *weakly essential* at p_0.

Theorem 1.4. *Let $M \subset \mathbb{C}^{n+1}$ be a real-analytic, connected hypersurface which is of 1-infinite type along a connected complex hypersurface $E \subset \mathbb{C}^{n+1}$ contained in M. Let $f \colon M \to \mathbb{C}^{n+1}$ be a C^∞-smooth CR mapping. Assume that $f(M)$ is contained in a real-analytic hypersurface $M' \subset \mathbb{C}^{n+1}$, and that*

- (i) *$f(M) \not\subset f(E)$;*
- (ii) *$f|_E \colon E \to \mathbb{C}^{n+1}$ is a finite mapping.*

Assume, in addition, that M' is weakly essential at some point of $f(E) \subset M'$. Then, f extends as a holomorphic mapping from an open neighborhood of E in $\mathbb{C}^{n+1}$, i.e. there exists a holomorphic mapping $F \colon U \to \mathbb{C}^{n+1}$, where U is an open neighborhood of E in $\mathbb{C}^{n+1}$, such that $F|_M = f$ (in $M \cap U$).

The conclusion of Theorem 1.4, under the additional assumption that f extends holomorphically to one side of M, was proved in [HMM00] for M of m-infinite type, with $1 \leq m < \infty$, and f merely assumed to be C^1-smooth.

We would also like to mention some results, related to those in this paper, without describing them in detail. The first is the Baouendi-Rothschild reflection principle [BR91] which deals with C^∞-smooth CR mappings which are assumed to extend to one side of a Levi-nonflat real-analytic hypersurface in $\mathbb{C}^2$, and the second is a general result ([BHR96]) for C^∞-smooth CR mappings of real algebraic, holomorphically nondegenerate submanifolds in any dimension. We also mention that there are some results (see e.g. [F89], [Han97], [Hay98], [La99]) for CR mappings between manifolds in spaces of different dimension.

The idea of the proof of our principal result, Theorem 1.4, is roughly the following. We first prolong the system (1), (2) (or, more precisely, its intrinsic counterpart) to a Pfaffian system on M. The latter system develops a singularity along the complex hypersurface $E \subset M$, and the nature of this singularity near points in general position on $E \subset M$ is described by the invariant m (see Theorem

2.1); when $m = 1$, the singularity is regular in a certain sense (Fuchsian), and one can use known results about such systems combined with the Hanges–Treves propagation theorem to complete the proof of Theorem 1.4. The details are carried out below.

2. SINGULAR PFAFFIAN SYSTEMS FOR CR MAPPINGS

As above, let M be a real-analytic hypersurface in $\mathbb{C}^{n+1}$ and $p_0 \in M$. We shall keep the notation introduced in the previous section. Thus, we assume that $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ are normal coordinates for M at $p_0 = (0, 0)$ and that M is defined by an equation of the form (3), where $\phi(z, \chi, s)$ is a holomorphic function satisfying (4). We also assume that M is of m-infinite type, for some $m \geq 1$, along the complex hypersurface E, where m is defined by (5). It is also shown in [Me95] that the lowest order r in the Taylor expansion

$$\phi_m(z, \chi) = \sum_{|\alpha|+|\beta| \geq r} c_{\alpha\beta} z^\alpha \chi^\beta, \tag{14}$$

where $\phi_m(z, \chi)$ is defined by (5), is a biholomorphic invariant of M at p_0; observe that (4) implies that $r \geq 2$. With these definitions of the integers m and r, we shall say that M is of *m-infinite type* r at $p_0 = (0, 0)$.

Let us write the defining equation of M as in (12). We shall also say that M is *m-infinite ℓ-nondegenerate* at $p_0 = (0, 0)$ if

$$\operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial^{|\alpha|}}{\partial \bar{z}^\alpha} \left(\frac{\partial \psi}{\partial z} \right) (0, 0, 0) \colon \forall |\alpha| \leq \ell \right\} = \mathbb{C}^n, \tag{15}$$

where $\partial/\partial z = (\partial/\partial z_1, \ldots, \partial/\partial z_n)$ and standard multi-index notation is used. The notion is completely analogous to that of ℓ-nondegeneracy for hypersurfaces of finite type (see e.g. [BER99], Chapter XI), and the reader can verify, as in the finite type case, that the definition above is independent of the choice of normal coordinates. We should point out, as is remarked in [HMM00], that if M is m-essential at some point on $E \subset M$, then M is m-infinite n-nondegenerate outside a proper complex subvariety of E.

Given a smooth manifold M, let us denote by $J^k(M, \mathbb{R}^N)_p$ the space of k-jets at $p \in M$ of smooth mappings $F \colon M \to \mathbb{R}^N$. Given a coordinate system $x = (x_1, \ldots, x_{2n+1})$ on M near p, there are natural coordinates $\lambda^k := (\lambda_i^\beta)$, where $1 \leq i \leq N$ and $\beta \in \mathbb{Z}_+^{2n+1}$ with $1 \leq |\beta| \leq k$, on $J^k(M, \mathbb{R}^N)_p$ in which the k-jet at p of a smooth mapping $F \colon M \to \mathbb{R}^N$ is given by $\lambda_i^\beta = (\partial_x^\beta F_i)(p)$, $1 \leq |\beta| \leq k$ and $1 \leq i \leq N$. The main technical result in this paper is the following.

Theorem 2.1. *Let $M, M' \subset \mathbb{C}^{n+1}$ be real-analytic hypersurfaces which are of m-infinite and m'-infinite type respectively, for some integers $m, m' \geq 1$, along complex hypersurfaces $E \subset M$ and $E' \subset M'$. Let $p_0 \in E \subset M$ and $p_0' \in E' \subset M'$. Assume that M' is m'-infinite ℓ-nondegenerate at $p_0' \in M$ and M is of m-infinite* *type* 2 *at* $p_0 \in M$*. Let* $f\colon M \to M' \subset \mathbb{C}^{n+1}$ *be a* C^∞*-smooth CR mapping which satisfies:*

(i) $f(M \setminus E) \subset M' \setminus E'$*;*
(ii) $f|_E\colon E \to \mathbb{C}^{n+1}$ *is a local immersion at* p_0*.*

Choose local coordinates $y = (x, s) \in \mathbb{R}^{2n} \times \mathbb{R}$ *on* M *vanishing at* p_0 *such that* E *is given by* $s = 0$*, and local coordinates* $y' = (x', s') \in \mathbb{R}^{2n} \times \mathbb{R}$ *on* M' *vanishing at* p'_0 *such that* E' *is given by* $s' = 0$*. Then, there are real-analytic functions* $r_i(y')(y)$*, for* $i = 1, \ldots, 2n$*, with the following property. If* $f(p_0)$ *is sufficiently close to* p'_0 *and we set* $h := s' \circ f$*,* $f_j := y'_j \circ f$*,* $g_i := x'_i \circ f$*,* $j = 1, \ldots, 2n+1$*, and* $i = 1, \ldots, 2n$*, then there is a* C^∞*-smooth function* $v = v_f$ *near* $p_0 = (0, 0)$ *such that*

$$s^m \partial_s h = v h^{m'} + \sum_{i=1}^{2n} r_i(f) s^m \partial_s g_i. \tag{16}$$

Moreover, if for any such mapping we write $u = u_f$ *for the* $\mathbb{R}^{(2n+1)^2}$*-valued function* $u := (\partial_{x_i} f_j, s^m \partial_s g_i, v)$*, where* $v = v_f$*, and we fix one such mapping* f^0 *with* $f^0(p_0) = p'_0$ *then, for any multi-index* $(\alpha, p) \in \mathbb{Z}_+^{2n} \times \mathbb{Z}$ *with*

$$|\alpha| + p = 2\ell + 1, \tag{17}$$

there is a real-analytic function $r^{\alpha,p}(\lambda^k; y')(y)$ *on* U*, where* $k := 2\ell$ *and* $U \subset J^k(M, \mathbb{R}^{(2n+1)^2})_{p_0} \times M' \times M$ *is an open neighborhood of* $(((s^m\partial_s)^q \partial_x^\beta u_{f^0})(p_0), f^0(p_0), p_0)$*, such that*

$$(s^m \partial_s)^p \partial_x^\alpha u = r^{\alpha,p}((s^m\partial_s)^q \partial_x^\beta u; f), \tag{18}$$

where $0 \le |\beta| + q \le k$*, for every smooth CR mapping* $f\colon V \to M'$*, where* $V \subset M$ *is some open neighborhood of* p_0*, which satisfies* (i)–(ii) *above and with* $(((s^m\partial_s)^q \partial_x^\beta u)(p_0), f(p_0), p_0) \in U$*; here,* u *denotes* u_f *as defined above. The functions* $r^{\alpha,p}$ *depend only on* M*,* M' *and the* k*-jet of* u_{f^0} *at* p_0*. In addition, the functions* $r^{\alpha,p}$ *are rational in* $\lambda^k \in J^k(M, \mathbb{R}^{(2n+1)^2})_{p_0}$*.*

Before turning to the proof of Theorem 2.1, we mention that a similar result was proved at minimal points (i.e. $m = 0$) in [Hay98], using ideas of [Han83], [Han97], and in [E00], using a more intrinsic approach which allowed its extension to the case of merely smooth hypersurfaces. In this paper, we shall follow the latter approach. As a consequence, Theorem 2.1 also follows with C^∞-smoothness (for the hypersurfaces M and M' as well as for the functions $r_j^{\alpha,p}$) replacing real-analyticity. However, we shall not pursue the case of merely smooth hypersurfaces in this paper.

3. Preliminaries

The proof of Theorem 2.1 is similar to that of Theorem 2 in [E00]. Indeed, the proof of Theorem 2.1 will essentially reduce to that in [E00] by suitably modifying and adapting the setup in [E00] to the present situation where the hypersurfaces are of infinite type. We first need to relate the notions of m-infinite type and m-infinite ℓ-nondegeneracy defined in the previous section to the CR geometry of M.

Recall that $\mathcal{V} := \mathbb{C}TM \cap T^{0,1}\mathbb{C}^{n+1}$ denotes the CR bundle on M. We shall denote by T^0M the characteristic bundle $(\mathcal{V} \oplus \bar{\mathcal{V}})^\perp \subset \mathbb{C}T^*M$, and by $T'M$ the holomorphic bundle $\mathcal{V}^\perp \subset \mathbb{C}T^*M$. Real non-vanishing sections of T^0M are called characteristic forms, sections of $T'M$ holomorphic forms, and sections of $\mathcal{V}$ CR vector fields. The reader is referred to [BER99] for basic definitions and facts regarding CR manifolds and structures.

The Levi form of M at $p \in M$ is a multi-linear mapping $\Lambda_p \colon \mathcal{V}_p \times \bar{\mathcal{V}}_p \times T^0_pM \to \mathbb{C}$ (or, equivalently, a tensor in $\mathcal{V}^*_p \times \bar{\mathcal{V}}^*_p \times (T^0_pM)^*$) defined by

$$\Lambda_p(X_p, Y_p, \theta_p) := \frac{1}{2i}\left\langle \theta, [X,Y] \right\rangle_p = -\frac{1}{2i}\left\langle d\theta, X \wedge Y \right\rangle_p, \tag{19}$$

where $X \in \Gamma(M, \mathcal{V})$ and $Y \in \Gamma(M, \bar{\mathcal{V}})$ are vector fields extending X_p and Y_p, respectively, and θ a characteristic form extending θ_p. We shall assume that M is of m-infinite type along a complex hypersurface $E \subset M$. We first claim that m is also the order of vanishing of the Levi form Λ (as a function of $p \in M$ with values in $\mathcal{V}^*_p \times \bar{\mathcal{V}}^*_p \times (T^0_pM)^*$) along E. More precisely, we choose, near a given point $p_0 \in E \subset M$, a basis for the CR vector fields $L_{\bar{1}}, \ldots, L_{\bar{n}}$, set $L_A := \overline{L_{\bar{A}}}$, choose a nonvanishing characteristic form θ, and represent Λ by the $n \times n$ matrix $(h_{\bar{A}B})$, $1 \le A, B \le n$, where

$$h_{\bar{A}B}(p) := -2i\Lambda_p(L_{\bar{A}}, L_B, \theta). \tag{20}$$

Then the following holds.

Proposition 3.1. *If $M \subset \mathbb{C}^{n+1}$ is a real-analytic hypersurface which is of m-infinite type along a complex hypersurface $E \subset M$ through $p_0 \in M$, then there exists a real-analytic, $n \times n$ matrix valued function $(h^0_{\bar{A}B})$, $1 \le A, B \le n$, in a neighborhood of p_0 such that the restriction $(h^0_{\bar{A}B})|_E$ is not identically 0, and*

$$h_{\bar{A}B} = \delta^m h^0_{\bar{A}B},$$

where δ denotes the distance (in the Riemannian metric on M inherited from the ambient space) of p to E and $h_{\bar{A}B}$ is an $n \times n$ matrix representing the Levi form as explained above. In addition, M is of m-infinite type 2 at p_0 if and only if $h^0_{\bar{A}B}(p_0) \neq 0$ for some $1 \le A, B \le n$.

Before proving Proposition 3.1, we shall introduce a special choice of basis for the CR vector fields on M near p_0. Let (z, w) be normal coordinates for M at p_0, so that M is defined near $p_0 = (0, 0)$ by (3). We may then take $(z, s) \in \mathbb{C}^n \times \mathbb{R}$, with $s = \text{Re } w$, to be local coordinates on M near $(0,0)$, and choose

$$L_{\bar{A}} = \partial_{\bar{z}_A} - \frac{i\phi_{\bar{z}_A}}{1+i\phi_s}\partial_s, \tag{21}$$

where $\phi = \phi(z, \bar{z}, s)$, and where we have used the notation $\partial_{\bar{z}_A} = \partial/\partial \bar{z}_A$ and $\phi_s = \partial\phi/\partial s$, etc. Observe for future reference that the vector fields L_A and L_B, for any $1 \leq A, B \leq n$, commute. In the coordinates (z, s), the distance δ is comparable to s. We also denote by T the vector field ∂_s, so that $T, L_1, \ldots, L_n, L_{\bar{1}}, \ldots, L_{\bar{n}}$ spans $\mathbb{C}TM$ near p_0. In addition, we choose the characteristic form θ so that $\langle \theta, T\rangle = 1$.

Proof of Proposition 3.1. It is not difficult to see that the order of vanishing of the Levi form along E is independent of the choice of basis $L_{\bar{1}}, \ldots, L_{\bar{n}}$ of the CR vector fields, and characteristic form θ near p_0. Thus, it suffices to prove Proposition 3.1 using the special choices introduced above. Since M is assumed to be of m-infinite type along E, we may write

$$\phi(z, \bar{z}, s) = s^m(\alpha(z, \bar{z}) + O(|z|^{r+1})) + O(s^{m+1}),$$

where $\alpha(z, \bar{z}) \not\equiv 0$ is a homogeneous polynomial of some degree $r \geq 2$ with $\alpha(z, 0) \equiv \alpha(0, \bar{z}) \equiv 0$. In particular, the $n \times n$ matrix $(\alpha_{\bar{z}_A z_B})$ is not identically 0. A straightforward calculation shows that

$$h_{\bar{A}B} = s^m(\alpha_{\bar{z}_A z_B}(z, \bar{z}) + O(|z|^{r-1})) + O(s^{m+1}),$$

which completes the proof of Proposition 3.1. □

Next, for $A_1, \ldots, A_k \in \{1, \ldots, n\}$, we define, following [E98], [E00], the functions

$$h_{\bar{A}_1\ldots\bar{A}_k D} := \left\langle \mathcal{L}_{\bar{A}_k} \ldots \mathcal{L}_{\bar{A}_1}\theta, L_D \right\rangle, \tag{22}$$

where $\mathcal{L}_{\bar{A}}\omega := \mathcal{L}_{L_{\bar{A}}}\omega$, for a holomorphic form ω, denotes the Lie derivative $L_{\bar{A}} \lrcorner d\omega$ along the CR vector field $L_{\bar{A}}$ (which is again a holomorphic form). It is shown in [E00] that

$$h_{\bar{A}_1\ldots\bar{A}_k\bar{C}D} = L_{\bar{C}} h_{\bar{A}_1\ldots\bar{A}_k D} + h_{\bar{A}_1\ldots\bar{A}_k} h_{\bar{C}D}, \tag{23}$$

where

$$h_{\bar{A}_1\ldots\bar{A}_k} := \left\langle \mathcal{L}_{\bar{A}_k} \ldots \mathcal{L}_{\bar{A}_1}\theta, T \right\rangle.$$

By using Proposition 3.1 and (23) inductively, we conclude that

$$h_{\bar{A}_1\ldots\bar{A}_k D} = s^m h^0_{\bar{A}_1\ldots\bar{A}_k D}, \tag{24}$$

where $h^0_{\bar{A}_1\ldots\bar{A}_k D}$ is a real-analytic function satisfying the identity

$$h^0_{\bar{A}_1\ldots\bar{A}_k\bar{C}D} = L_{\bar{C}} h^0_{\bar{A}_1\ldots\bar{A}_k D} + a_{\bar{C}} h^0_{\bar{A}_1\ldots\bar{A}_k D} + h_{\bar{A}_1\ldots\bar{A}_k} h^0_{\bar{C}D}; \tag{25}$$

here, $a_{\bar{C}}$ is the function $m(L_{\bar{C}}s)/s$, which is real-analytic since $L_{\bar{C}}s$ vanishes on E.

Now, define the filtration

$$\bar{\mathcal{V}}_0 = F_0(0) \supset F_1(0) \supset \ldots \supset F_k(0) \supset \ldots \supset \{0\}, \tag{26}$$

where each subspace $F_k(0)$ is defined as follows,

$$F_k(0) := \left\{ X_0 = a^D L_D(0) \in \bar{\mathcal{V}}_0 \colon a^D h^0_{\bar{A}_1 \ldots \bar{A}_j D}(0) = 0,\ \forall A_1 \ldots A_j \in \{1, \ldots n\},\ j \leq k \right\}. \tag{27}$$

In (27), we have used the summation convention, i.e. an index appearing both as a super- and subscript is summed over. Moreover, in what follows, capital Roman indices (A, B, etc.) will run over the set $\{1, \ldots, n\}$. As in [E98], one can check that

$$(X_1, \ldots, X_k, Y, \theta) \to \frac{1}{s^m} \lim_{(z,s)\to(0,0)} \langle \mathcal{L}_{X_k} \ldots \mathcal{L}_{X_1}\theta, Y \rangle\,,$$

defines a multi-linear mapping

$$\underbrace{\mathcal{V}_0 \times \ldots \times \mathcal{V}_0}_{k \text{ times}} \times F_{k-1}(0) \times T^0_0 M \to \mathbb{C},$$

which is symmetric in the first k positions. Set $r_k = n - \dim F_k(0)$. By a constant linear change of the L_A, we may adapt the basis L_A of the complex conjugate CR vector fields to the filtration (26) so that $L_{r_k+1}(0), \ldots L_n(0)$ spans $F_k(0)$ for each $k = 0, 1, \ldots, \ell$, where ℓ is the smallest integer for which $F_\ell(0)$ is minimal. We also adopt the index convention from [E00]. For $j = 1, 2, \ldots$, Greek indices $\alpha^{(j)}, \beta^{(j)}$, etc., will run over the set $\{1, \ldots, r_{j-1}\}$ and small Roman indices $a^{(j)}, b^{(j)}$, etc., over $\{r_{j-1}+1, \ldots, n\}$. As mentioned above, capital Roman indices A, B, etc., will run over $\{1, \ldots, n\}$. The linear change of the L_A which adapts the basis to the filtration (26) corresponds to a linear change of the coordinates z in normal coordinates, so that we may still assume that the basis $L_{\bar{A}}$ is of the form (21).

A straightforward calculation in normal coordinates (cf. [E97] and [BER99] for the calculation in the finite type case) shows that M is m-infinite ℓ-nondegenerate if and only if $F_\ell(0) = \{0\}$ and $F_j(0) \neq \{0\}$ for $j < \ell$. This is equivalent to the fact that there are indices $\underline{A}^j := A^j_1 \ldots A^j_{k_j}$, $j = 1, \ldots, n$ with $k_j \leq \ell$, such that the $n \times n$ matrix $(h^0_{\underline{A}^j D}(0))$, $1 \leq j, D \leq n$ is invertible. In addition, by the choice of basis L_A and the index convention, we have the following two basic facts, whose proofs are elementary and left to the reader (c.f. also [E00]). First, we have

$$h^0_{\bar{A}_1 \ldots \bar{A}_j a^{(k)}}(0) = 0, \quad \forall j < k, \tag{28}$$

and, secondly,

Lemma 3.2. *If* $(v^A) \in \mathbb{C}^n$ *satisfies*

$$v^{a^{(k)}} h^0_{\bar{A}_1 \ldots \bar{A}_k a^{(k)}}(0) = 0, \quad \forall A_1 \ldots A_k \in \{1, \ldots, n\},$$

for some $k \leq \ell$, *then* $v^{\alpha^{(k+1)}} = 0$.

For future reference, we also record here the fact that any commutator $[X,Y]$, for $X, Y \in \{T, L_A, L_{\bar A}\}$, is a multiple of T, and a straightforward calculation shows that

$$
\begin{aligned}
[L_{\bar A}, s^m T] &= m s^{m-1}(L_{\bar A} s)T + s^m [L_{\bar A}, T] \\
&= -s^m h^0_{\bar A},
\end{aligned}
\tag{29}
$$

where $h^0_{\bar A}$ is a real-analytic function near 0 since $L_{\bar A} s$ vanishes on E.

4. Proof of Theorem 2.1

We shall keep the notation and conventions introduced in previous sections for the real-analytic hypersurface $M \subset \mathbb{C}^{n+1}$. We shall also need the real vector field $S = s^m T$, where m is the invariant associated to M in Theorem 2.1; i.e. M is of m-infinite type along E. For convenience of notation, we shall denote the target hypersurface $M' \subset \mathbb{C}^{n+1}$ by $\hat M$ and denote corresponding objects for $\hat M$ by placing a hat over them; i.e. $\hat E$ denotes the complex hypersurface through $\hat p_0$ contained in $\hat M$, $(\hat z, \hat s) \in \mathbb{C}^n \times \mathbb{R}$ denote local coordinates on $\hat M$ near $\hat p_0 = (0,0)$, $\hat L_{\bar A}$ denotes a basis for the CR vector fields on $\hat M$ of the form (21), $\hat T$ denotes the real vector field $\partial/\partial \hat s$, etc.

Assume that $f\colon M \to \hat M$ is a smooth (C^∞) CR mapping defined near $p_0 = 0$ in M such that $f(0) = \hat p_0 = 0$ and f satisfies conditions (i)–(ii) in Theorem 2.1. Recall that a smooth mapping $f\colon M \to \hat M$ is called CR if $f_*(\mathcal{V}_p) \subset \hat{\mathcal{V}}_{f(p)}$, where $f_*\colon \mathbb{C}TM \to \mathbb{C}T\hat M$ denotes the tangent mapping or push forward, for every $p \in M$. When $E, \hat E \subset \mathbb{C}^{n+1}$ are complex hypersurfaces contained in M and $\hat M$, respectively, then any CR mapping $f\colon M \to \hat M$ sends E into $\hat E$. Thus, condition (ii) in Theorem 2.1 is equivalent to $f|_E\colon E \to \hat E$ being a local diffeomorphism. Also, observe that if $f|_E\colon E \to \hat E$ is a local diffeomorphism near 0 then $f_*(\mathcal{V}_0) = \hat{\mathcal{V}}_0$. We introduce the smooth $GL(\mathbb{C}^n)$-valued function (γ^A_B), smooth complex-valued functions η^A, and real-valued function ξ so that

$$
f_*(L_B) = \gamma^A_B \hat L_A, \quad f_*(L_{\bar B}) = \overline{\gamma^A_B} \hat L_{\bar A}, \quad f_*(S) = \xi \hat S + \eta^A \hat L_A + \overline{\eta^A} \hat L_{\bar A}.
\tag{30}
$$

Observe that ξ is *a priori* possibly singular along $f^{-1}(\hat E)$ since $\hat S$ vanishes along $\hat E$. Indeed, using the coordinates introduced in the previous section, we have

$$
s^m \frac{\partial \hat s}{\partial s} = \hat s^{\hat m} \xi + \frac{i s^m}{1 - i\hat\phi_s} \frac{\partial \hat\phi}{\partial \hat z_A} \frac{\partial \hat z_A}{\partial s} - \frac{i s^m}{1 + i\hat\phi_s} \overline{\frac{\partial \hat\phi}{\partial \hat z_A} \frac{\partial \hat z_A}{\partial s}}.
\tag{31}
$$

However, we shall show (Proposition 4.1) that if M is of m-infinite type 2 at 0, then ξ is in fact smooth near 0.

We can write (30) using matrix notation as

$$f_*(S, L_B, L_{\bar B}) = (\hat S, \hat L_A, \hat L_{\bar A}) \begin{pmatrix} \xi & 0 & 0 \\ \eta^A & \gamma^A_B & 0 \\ \overline{\eta^A} & 0 & \overline{\gamma^A_B} \end{pmatrix}. \tag{32}$$

Hence, if we let $\theta, \theta^A, \theta^{\bar A}$, where θ is as in section 3, be a dual basis (of 1-forms) to $T, L_A, L_{\bar A}$ then, by duality, we have (outside E in view of condition (i) in Theorem 2.1)

$$f^* \begin{pmatrix} \hat\theta/\hat s^{\hat m} \\ \hat\theta^A \\ \hat\theta^{\bar A} \end{pmatrix} = \begin{pmatrix} \xi & 0 & 0 \\ \eta^A & \gamma^A_B & 0 \\ \overline{\eta^A} & 0 & \overline{\gamma^A_B} \end{pmatrix} \begin{pmatrix} \theta/s^m \\ \theta^B \\ \theta^{\bar B} \end{pmatrix}. \tag{33}$$

We shall make use of the two identities

$$\langle df^*\hat\omega, X\wedge Y\rangle = \langle d\hat\omega, f_*X \wedge f_*Y\rangle, \tag{34}$$

where the left side is evaluated at $p \in M$ and the right side at $f(p)$, which holds for any 1-form $\hat\omega$ on $\hat M$ and vector fields X, Y on M, and also

$$\langle d\omega, X\wedge Y\rangle = -\langle \hat\omega, [X,Y]\rangle, \tag{35}$$

which holds for any 1-form $\omega \in \{\theta/s^m, \theta^A, \theta^{\bar A}\}$ and vector fields $X, Y \in \{S, L_A, L_{\bar A}\}$ on M since $\theta/s^m, \theta^A, \theta^{\bar A}$ is a dual basis (outside E) to $S, L_A, L_{\bar A}$. First, we apply (34) with $\hat\omega = \hat\theta/\hat s^{\hat m}$, $X = L_{\bar A}$, and $Y = L_B$, and obtain

$$\begin{aligned} \left\langle df^*(\hat\theta/\hat s^{\hat m}), L_{\bar A}\wedge L_B\right\rangle &= \left\langle d(\hat\theta/\hat s^{\hat m}), f_*L_{\bar A}\wedge f_*L_B\right\rangle \\ &= \overline{\gamma^C_A}\gamma^D_B \left\langle d(\hat\theta/\hat s^{\hat m}), \hat L_{\bar C}\wedge \hat L D\right\rangle \\ &= -\overline{\gamma^C_A}\gamma^D_B(\hat h^0_{\bar C D}\circ f), \end{aligned} \tag{36}$$

where the last identity follows from (35) and Proposition 3.1. On the other hand, by (33), we have

$$\begin{aligned} \left\langle df^*(\hat\theta/\hat s^{\hat m}), L_{\bar A}\wedge L_B\right\rangle &= \langle d(\xi\theta/s^m), L_{\bar A}\wedge L_B\rangle \\ &= -\xi h^0_{\bar A B}, \end{aligned} \tag{37}$$

where again the last identity follows from (35) and Proposition 3.1. Thus, we have the identity

$$\xi h^0_{\bar A B} = \gamma^D_B \overline{\gamma^C_A}\hat h^0_{\bar C D}. \tag{38}$$

Here, and in what follows, we abuse the notation in the following way. For a function $\hat c$ defined on $\hat M$, we use the notation $\hat c$ to denote both the function $\hat c \circ f$ on M and the function $\hat c$ on $\hat M$. It should be clear from the context which of the two functions is meant. For instance, in (38), we must have $\hat h^0_{\bar C D} = \hat h^0_{\bar C D}\circ f$.

By repeating the procedure above to the equation (34) with $\hat\omega = \hat\theta/\hat s^{\hat m}$, $X = L_{\bar A}$, and $Y = S$, we obtain

$$L_{\bar A}\xi + \xi h^0_{\bar A} = \xi\overline{\gamma^C_A}\hat h^0_{\bar C} + \overline{\gamma^C_A}\eta^D\hat h^0_{\bar C D}, \tag{39}$$

where $h^0_{\bar A}$ is the real-analytic function defined by (29). Next, applying (34) with $\hat\omega = \hat\theta^E$, $X = L_{\bar A}$, and $Y = L_B$, we obtain

$$L_{\bar A}\gamma^E_B + \eta^E h^0_{\bar A B} = 0, \tag{40}$$

by also using the facts that $[L_{\bar A}, L_B]$ and $[\hat L_{\bar C}, \hat L_D]$ are multiples of T and $\hat T$ respectively. Applying (34) with $\hat\omega = \hat\theta^E$, $X = L_{\bar A}$, and $Y = S$, we obtain

$$L_{\bar A}\eta^E + \eta^E h^0_{\bar A} = 0. \tag{41}$$

To obtain (41), we have used the facts that commutators of CR vector fields are CR vector fields, and that $[L_{\bar A}, S]$ and $[\hat L_{\bar C}, \hat S]$ are multiples of T and $\hat T$ respectively. Finally, we apply (34) with $\hat\omega = \hat\theta^E$, $X = S$, and $Y = L_A$ and obtain

$$S\gamma^E_A - L_A\eta^E - \eta^E\overline{h^0_{\bar A}} = 0. \tag{42}$$

Before proceeding, we observe the following important consequence of (38).

Proposition 4.1. *If $M \subset \mathbb{C}^{n+1}$ is of m-infinite type 2 at $0 \in M$, then the function ξ defined in (30) is smooth near 0.*

Proof. The conclusion follows immediately from (38) and Proposition 3.1. □

Equations (38)–(42) are completely analogous to (2.9)–(2.13) in [E00]. By following the arguments in that paper (essentially word for word), repeatedly applying the vector fields $L_{\bar A}$ to (38) and (39), we obtain the following reflection identities which are analogous to those in [E00], Theorem 2.4.

Theorem 4.2. *If $\hat M$ is $\hat m$-infinite ℓ-nondegenerate at $0 \in \hat M$, then the following identities hold for any indices $D, E \in \{1, \ldots, n\}$,*

$$\begin{aligned} \gamma^D_E &= r^D_E\big(\overline{L^J\gamma^C_A}, \overline{L^I\xi}; f\big), \\ \eta^D &= s^D\big(\overline{L^J\gamma^C_A}, \overline{L^I\xi}; f\big) \end{aligned} \tag{43}$$

where

$$r^D_E\big(\overline{L^J\gamma^C_A}, \overline{L^I\xi}; q\big)(p), \quad s^D\big(\overline{L^J\gamma^C_A}, \overline{L^I\xi}; q\big)(p) \tag{44}$$

are real-analytic functions which are rational in $\overline{L^J\gamma^C_A}$ and polynomial in $\overline{L^I\xi}$, the indices A, C run over the set $\{1, \ldots, n\}$, and J, I over all multi-indices with $|J| \le \ell - 1$ and $|I| \le \ell$; here, $(p, q) \in M \times \hat M$. Moreover, the functions in (44) depend only on M and $\hat M$ (and not on the mapping f).

To complete the proof of Theorem 2.1, we shall use the following result whose proof follows from that of [E00], Proposition 3.18.

Proposition 4.3. *If $M \subset \mathbb{C}^{n+1}$ is of m-infinite type 2 then, for any multi-index J, integer $k \geq 1$, and index $F \in \{1, \dots, n\}$ there exist real-analytic functions $b_q^{E_1 \dots E_j}$ such that*

$$\sum_{j=1}^{|J|+k} \sum_{q=0}^{k} b_q^{E_1 \dots E_j} \underbrace{[\dots[L_{E_1} \dots L_{E_j}, L_{\bar{1}}], L_{\bar{1}}] \dots, L_{\bar{1}}]}_{\text{length } q} = L^J S^k, \tag{45}$$

where standard multi-index notation is used and the length of the commutator $[\dots[X, Y_1], Y_2] \dots, Y_q]$ is q.

The arguments in [E00] (indeed, with the simplification described in the remark following the proof of Theorem 2 due to the assumption that M is of m-infinite type 2 at 0) with T replaced by S now shows that for any multi-indices R and Q, any nonnegative integer k, and any indices $D, F \in \{1, \dots n\}$, there are smooth functions, which are rational in their arguments preceeding the ";", such that

$$\begin{aligned} L^R S^k L^{\bar{Q}} \gamma_F^D &= r^{R\bar{Q}k}\big(L^I S^j \gamma_A^C, L^I S^j \eta^C, L^I S^j \xi; f\big), \\ L^R S^k L^{\bar{Q}} \eta_F^D &= s^{R\bar{Q}k}\big(L^I S^j \gamma_A^C, L^I S^j \eta^C, L^I S^j \xi; f\big) \\ L^R S^k L^{\bar{Q}} \xi &= t^{R\bar{Q}k}\big(L^I S^j \gamma_A^C, L^I S^j \eta^C, L^I S^j \xi, \overline{L^I S^j \gamma_A^C}, \overline{L^I S^j \eta^C}, \overline{L^I S^j \xi}; f\big) \end{aligned} \tag{46}$$

where $|I| + j \leq 2\ell$. The conclusion of Theorem 2.1 follows by writing (46), for all R, Q, k such that

$$|R| + |Q| + k = 2\ell + 1$$

in the coordinate system (x, s), where $x = (\mathrm{Re}\ z_1, \mathrm{Im}\ z_1, \dots, \mathrm{Re}\ z_n, \mathrm{Im}\ z_n)$, for M near 0 and the analogous coordinate system $\hat{y} = (\hat{x}, \hat{s})$ for $\hat{M}$ near $0 \in \hat{M}$, and observing that the same system of differential equations holds for any other CR mapping f' sending a neighborhood of 0 in M into $\hat{M}$ with $(((s^m \partial_s)^q \partial_x^\beta u_{f'})(0), f'(0))$ sufficiently close to $(((s^m \partial_s)^q \partial_x^\beta u_f)(0), f(0))$, where $|\beta| + q \leq k := 2\ell$ and the notation u_f is as in Theorem 2.1. This completes the proof of Theorem 2.1. ☐

5. PROOFS OF THEOREMS 1.1 AND 1.4

Proof of Theorem 1.4. By the Hanges–Treves propagation theorem (see [HT83]), it suffices to show that f extends holomorphically to a full neighborhood of some particular point $p_1 \in E \subset M$. We shall first choose a point $p_0 \in E$ so that Theorem 2.1 is applicable. First, since M is of 1-infinite type along E, M is in fact of 1-infinite type 2 outside a proper real-analytic variety of E in view of Proposition 3.1. Also, since $f|_E \colon E \to \mathbb{C}^{n+1}$ is finite and $f(E) \subset M'$, $E' := f(E)$ is a connected complex hypersurface contained in M'. By assumption, M' is m'-essential at some point of E', for some integer $m' \geq 1$. (It is not difficult to see that, in fact, m' has to be one.) It follows, exactly as in the finite type case (see e.g. [BER99], Chapter IX; cf. also the arguments in [HMM00]) that M' is m'-infinite n-nondegenerate outside a proper real-analytic subvariety of E'. Since $f|_E$

is finite, we can find $p_0 \in E$ such that $f|_E \colon E \to E'$ is a local biholomorphism, M' is m'-infinite n-nondegenerate at $p_0' := f(p_0)$, and, in addition, M is of 1-infinite type 2 at p_0. To apply Theorem 2.1, we must show that, for some small open neighborhood U of p_0 in M, $f(U \setminus E) \subset M' \setminus E'$. We may choose U so that $U \setminus E$ has two connected components. Assume, in order to reach a contradiction, that f maps one of these components into E'. Then the component of f which is transversal to E' must vanish to infinite order along E. This implies, by [E96, Theorem 4.4], that f maps U into E' which contradicts condition (i) of Theorem 1.4. Hence, $f(U \setminus E) \subset M' \setminus E'$ and we may apply Theorem 2.1 to the mapping $f^0 = f$ at p_0 with $m = 1$ and $\ell = n$.

Let us choose local coordinates $y = (x, s) \in \mathbb{R}^{2n} \times \mathbb{R}$, vanishing at p_0, on M and $y' = (x', s') \in \mathbb{R}^{2n} \times \mathbb{R}$, vanishing at p_0', on M' as described in Theorem 2.1. Following the notation in that theorem (with $m = 1$), we denote by $f_j := y_j' \circ f$ (note that here f_j does not mean a component of the mapping $f \colon M \to \mathbb{C}^{n+1}$), $g_l := x_l' \circ f$, $h := s' \circ f$, and also $u := (\partial_{x_l} f_j, s\partial_s g_l, v)$; here $v = v_f$ is as defined in Theorem 2.1, $j \in \{1, \ldots, 2n+1\}$, and $l \in \{1, \ldots, 2n\}$. We shall also write, for each $i \in \{1, \ldots, (2n+1)^2\}$, each multi-index $\alpha \in \mathbb{Z}_+^{2n}$ and each non-negative integer p,

$$u_i^{\alpha,p} := (s\partial_s)^p \partial_x^\alpha u_i$$

and also U for the vector $(f_j, u_i, u_i^{\beta,q})$, where $|\beta| + q \le k := 2n$, $j \in \{1, \ldots, 2n+1\}$, and $i \in \{1, \ldots, (2n+1)^2\}$. Hence, by Theorem 2.1, we have

$$(s\partial_s) u_i^{\alpha,p} = r_i^{\alpha,p}(U), \tag{47}$$

for each $i \in \{1, \ldots, (2n+1)^2\}$ and each multi-index α and non-negative integer p such that $|\alpha| + p = k$. If we add the contact equations

$$(s\partial_s) u_i^{\alpha,p} = u_i^{\alpha,p+1}, \tag{48}$$

for $|\alpha| + p < k$, and the equations

$$(s\partial_s) g_l = u_{2n(2n+1)+l}, \quad (s\partial_s) h = h^{m'} u_{(2n+1)^2} + \sum_{l=1}^{2n} r_l(f) u_{2n(2n+1)+l}, \tag{49}$$

then we obtain the system

$$(s\partial_s) U = R(U), \tag{50}$$

where $R(U)(x, s)$ is a real-analytic vector valued function of U and (x, s), depending only on M near $p_0 = (0, 0)$ and M' near $p_0' = 0$ (and the possibly the value of $U(0, 0)$).

Let us fix $x \in \mathbb{R}^{2n}$ near 0 and consider (50) as a system of ordinary differential equations for $U(x, \cdot)$. This system has a singularity of so-called Briot–Bouquet type at $s = 0$, and its properties are well understood (see e.g. [S91], Chapters 3.6–3.7 and 8.8; or [Hi76] and further references in these books). We shall use the following result, which is undoubtedly known. However, the author has not found a satisfactory reference for it and, hence, we shall provide a proof.

Theorem 5.1. *Let $y(t)$, with $y = (y_1, \ldots, y_N)$ and $t \in \mathbb{R}$, be a C^∞-smooth $\mathbb{C}^N$-valued function near $t = 0$ such that $y(0) = 0$ and*

$$t\,\frac{dy_j}{dt} = f_j(t, y), \quad j = 1, \ldots, N, \tag{51}$$

where the $f_j(t, y)$ are analytic functions near $(t, y) = (0, 0)$. Then, $y(t)$ is real-analytic near 0.

Proof. A classical result due to Malmquist ([Ma21]; cf. also [Hi76], Chapter 11.1) states that the general solution $y(t) = y_1(t)$ of (51), with $N = 1$ and $\lim_{t\to 0} y(t) = 0$, is given for $t > 0$ by a convergent series of the form

$$y(t) = \sum_{j=0}^{\infty} \sum_{r=0}^{r_j} c_{ljr} (\ln t)^r t^{\nu_j}, \tag{52}$$

where the r_j are integers, and $0 < \nu_0 < \nu_1 < \ldots < \nu_j \nearrow \infty$. A similar convergent series represents $y(t)$ for $t < 0$. If we assume that the solution $y(t)$ is C^∞-smooth near $t = 0$, then no fractional powers or logarithmic terms can appear and the series for $t < 0$ must match that for $t > 0$. We conclude that $y(t)$ is given by a convergent power series in t and, hence, $y(t)$ is real-analytic near $t = 0$. This completes the proof of Theorem 5.1 in the special case $N = 1$. (A similar result for arbitrary N is implicit in the literature, but the author has been unable to find an explicit reference for such a result.)

To treat the case of a general N, we shall use a result by Dulac (building on an idea of Poincare) reducing the system (51) to a special form. After an invertible linear transformation of the y_j if necessary, we may assume that $f = (f_1, \ldots, f_N)$ is of the form

$$f(t, y) = pt + Ay + O(2), \tag{53}$$

where $p \in \mathbb{C}^N$, A is an $N \times N$-matrix in Jordan normal form, and $O(2)$ as usual denotes terms of at least order two in all the variables. We shall denote the eigenvalues of A, repeated with multiplicity, by $\lambda_1, \ldots, \lambda_N$. The solution curve $t \to (t, y(t)) \in \mathbb{C} \times \mathbb{C}^N$ satisfies the differential system

$$\frac{dt}{t} = \frac{dy_1}{f_1(t, y)} = \ldots = \frac{dy_N}{f_N(t, y)}. \tag{54}$$

The characteristic roots of this system are $1, \lambda_1, \ldots, \lambda_N$. By a theorem of Dulac ([D12]) there are an integer p (determined by the location of the roots $1, \lambda_1, \ldots, \lambda_N$ in the complex plane), with $0 \le p \le N$, and analytic functions $\Phi_j(t, z_1, \ldots, z_p)$, $j = 1, \ldots, p$, $\Psi_i(t, z_1, \ldots, z_p)$, $i = 1, \ldots, N - p$, with $\Phi_j(t, z) = z_j + O(2)$ and

$\Psi_i(t,z) = O(2)$ such that

$$\begin{aligned} y_j &= \Phi_j(t,z), \quad j = 1, \ldots, p, \\ y_{p+i} &= \Psi_i(t,z), \quad i = 1, \ldots, N-p \end{aligned} \tag{55}$$

and the system (54), provided that $p \geq 1$, pulled back to the (t,z)-space, becomes

$$\frac{dt}{t} = \frac{dz_1}{g_1(t,z)} = \ldots = \frac{dz_p}{g_p(t,y)}, \tag{56}$$

where

$$g_j(t,z) = \lambda_j z_j + P_j(t, z_1, \ldots, z_{j-1}), \quad j = 1, \ldots, p, \tag{57}$$

and P_j are polynomials which do not involve the variables $z_j, \ldots, z_p$. (If $p = 0$ (which corresponds to all eigenvalues $\lambda_1, \ldots, \lambda_N$ being located on the negative real line $\mathbb{R}_- \cup \{0\}$), then there are no variables z in (55). This already implies that the $y_i(t)$, $i = 1, \ldots, N$ are real-analytic. Thus, in what follows we may assume that $p \geq 1$.) Observe that the p first equations in (55) can be solved for $z_j = \Theta_j(t, y_1, \ldots, y_p)$, $j = 1, \ldots, p$. Hence, by substituting the smooth functions $y = y(t)$ in this identity, we conclude that $z = z(t)$ is a C^∞-smooth $\mathbb{C}^p$-valued function that satisfies the system

$$t\,\frac{dz_j}{dt} = g_j(t,y), \quad j = 1, \ldots, p, \tag{58}$$

where g_j are as in (57). By applying the theorem of Malmquist cited above to the equation for z_1 (which is a single equation of Briot-Bouquet type for $z_1(t)$), we conclude that $z_1(t)$ is real-analytic near $t = 0$. By substituting the real-analytic function $z_1(t)$ into the equation for $z_2(t)$, using the "triangular" form of the equations (58), and again applying Malmquist's theorem, we conclude that $z_2(t)$ is real-analytic. By repeating this procedure inductively, we conclude that all the $z_1(t), \ldots, z_p(t)$ are real-analytic near 0. The real-analyticity of $y(t)$ now follows from (55). This completes the proof of Theorem 5.1. ☐

By applying Theorem 5.1 for fixed x (possibly after subtracting the value $U(x,0)$ from $U(x,s)$), we conclude that $U(x,\cdot)$ is given by a convergent series

$$U_l(x,s) = \sum_{j \geq 0} a_{lj}(x) s^j. \tag{59}$$

Moreover, the smoothness of U also implies that the coefficients $a_{lj}(x)$ are smooth functions of x. A simple Baire category argument shows that there are $x^1 \in \mathbb{R}^{2n}$ near 0, $\epsilon > 0$, and $\delta > 0$ such that the series (59) converge uniformly in x, with $|x - x^1| \leq \epsilon$, for $|s| \leq \delta$. It is well known (see e.g. [BER99], Proposition 1.7.5) that this implies that the CR mapping f extends holomorphically to a full neighborhood of $p_1 = (x^1, 0) \in M$ in $\mathbb{C}^{n+1}$. This completes the proof of Theorem 1.4, in view of the remark at the beginning of the proof. ☐

Proof of Theorem 1.1. The proof of Theorem 1.1 will follow from Theorem 1.4 if we can show that M' is necessarily weakly essential at some point of $E' := f(E)$. Exactly as in the finite type case, one can easily show that in $\mathbb{C}^2$ being m'-essential, for some integer m', at some point is equivalent to being of m'-infinite type r' for some integer r' (cf. e.g. [EH00]). Also, as mentioned in the proof of Theorem 1.4 above, a real-analytic hypersurface M' is of m'-infinite type 2 on a dense set of $E' \subset M'$ if it is of m'-infinite type along E'. Hence, to prove Theorem 1.1 it suffices to show that M' is of m'-infinite type along E' for some integer m'; i.e. we must show that M' is not Levi flat ($m' = \infty$). But this follows easily from the fact that M is not Levi flat ($m = 1$) and condition (i) (see e.g. [E96], Theorem 2.2). This completes the proof of Theorem 1.1. ☐

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