UNIFORMLY LEVI DEGENERATE CR MANIFOLDS; THE 5 DIMENSIONAL CASE

PETER EBENFELT¹

ABSTRACT. In this paper, we consider real hypersurfaces M in $\mathbb{C}^3$ (or more generally, 5-dimensional CR manifolds of hypersurface type) at uniformly Levi degenerate points, i.e. Levi degenerate points such that the rank of the Levi form is constant in a neighborhood. We also require the hypersurface to satisfy a certain second order nondegeneracy condition (called 2-nondegeneracy) at the point. One of our main results is the construction, near any point $p_0 \in M$ satisfying the above conditions, of a principal bundle $P \rightarrow M$ and a $\mathbb{R}^{\dim P}$ -valued 1-form ω , uniquely determined by the CR structure on M , which defines an absolute parallelism on P . If M is real-analytic, then covariant derivatives of ω yield a complete set of local biholomorphic invariants for M . This solves the biholomorphic equivalence problem for uniformly Levi degenerate hypersurfaces in $\mathbb{C}^3$ at 2-nondegenerate points.

0. INTRODUCTION

0.1. A brief history and basic concepts; motivation for the present work. A fundamental problem in the study of real submanifolds in complex space is the *biholomorphic equivalence problem* which in its most general form asks for (intrinsic) conditions on two submanifolds $M, M' \subset \mathbb{C}^N$ at distinguished points $p_0 \in M$, $p'_0 \in M'$ which guarantee that there exists a local biholomorphism $H: \mathbb{C}^N \rightarrow \mathbb{C}^N$ defined near p_0 such that $H(p_0) = p'_0$ and $H(M \cap U) = M' \cap U'$, for some open neighborhoods $U, U' \subset \mathbb{C}^N$ of p_0 and p'_0 respectively. When M and M' are real-analytic, an equivalent formulation is to ask for a real-analytic local CR diffeomorphism $f: M \rightarrow M'$ defined near $p_0 \in M$ with $f(p_0) = p'_0$. (For standard definitions and results on real submanifolds in complex space and abstract CR structures, the reader is referred e.g. to [BER]).

The case where M and M' are real-analytic and Levi nondegenerate hypersurfaces was solved by Cartan [C1–2] in $\mathbb{C}^2$, and by Tanaka [T1–2] and Chern–Moser [CM] in $\mathbb{C}^N$, $N \geq 2$. The solution consists of producing a fiber bundle $Y \rightarrow M$, for any given Levi nondegenerate hypersurface $M \subset \mathbb{C}^N$, and 1-form ω on Y , valued in $\mathbb{R}^{\dim Y}$, which at every $y \in Y$ gives an isomorphism between $T_y Y$ and $\mathbb{R}^{\dim Y}$ (an absolute parallelism or $\{1\}$ -structure on Y ; see e.g. [KN] or [K]) such that the following holds. If there exists a CR diffeomorphism $f: M \rightarrow M'$, then there exists

1991 *Mathematics Subject Classification.* 32F25, 32F40.

¹Supported in part by a grant from the Swedish Natural Science Research Council.

a diffeomorphism $F: Y \rightarrow Y'$ (where corresponding objects for M' are denoted with $'$) such that $F^*\underline{\omega}' = \underline{\omega}$ and the following diagram commutes

$$(0.1.1) \quad \begin{array}{ccc} Y & \xrightarrow{F} & Y' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{f} & M'. \end{array}$$

Conversely, if there exists a diffeomorphism $F: Y \rightarrow Y'$ such that $F^*\underline{\omega}' = \underline{\omega}$, then there exists a CR diffeomorphism $f: M \rightarrow M'$ such that (0.1.1) commutes. Suppose that such a bundle $Y \rightarrow M$ and $\mathbb{R}^{\dim Y}$ -valued 1-form $\underline{\omega}$ can be constructed for every M in some given class of manifolds. Then we shall say that the bundle $Y \rightarrow M$ with 1-form $\underline{\omega}$ *reduces the CR structure on M to a parallelism* (in this class). The construction of a bundle $Y \rightarrow M$ which reduces the CR structure on M to a parallelism for a class of CR submanifolds in $\mathbb{C}^N$ reduces the biholomorphic equivalence problem for the real-analytic manifolds in this class to the equivalence problem for $\{1\}$ -structures. The latter problem was solved by Cartan, and is well understood (see e.g. [G] or [S]).

The bundle $Y \rightarrow M$ constructed in [CM] is in fact a principal fiber bundle with group G_0 , where G_0 is the isotropy subgroup of $SU(p+1, q+1)$, $p+q = N-1$, and p, q are the number of positive and negative eigenvalues, respectively, of the Levi form. The authors of [CM] also construct a Cartan connection Π valued in the Lie algebra $\mathfrak{su}(p+1, q+1)$ which defines the same parallelism as $\underline{\omega}$ (given a suitable identification of $\mathbb{R}^{\dim Y}$ with $\mathfrak{su}(p+1, q+1)$). Covariant differentiation of the curvature $\Omega := d\Pi - \Pi \wedge \Pi$ produces a complete set of invariants for a real-analytic Levi nondegenerate hypersurface. In particular, it follows that a real-analytic strongly pseudoconvex hypersurface in $\mathbb{C}^N$ is locally biholomorphic to a piece of the $(2N-1)$ -sphere (the “standard model” for such hypersurfaces) if and only if the curvature Ω is identically zero (i.e. the connection is flat). The reader is also referred to the work of Burns-Shnider [BS] and Webster [We1] for further discussion in the Levi nondegenerate case.

More recently, CR manifolds of higher codimension whose Levi forms are suitably nondegenerate have been studied by several authors. Since the main focus in the present paper is on hypersurfaces, we mention only the papers by Čap–Schichl [CS], Ezhov–Isaev–Schmalz [EIS], Garrity–Mizner [GM], Schmalz–Slovák [SS], and refer the interested reader to these papers for further information about the higher codimensional case.

In this paper, we consider real hypersurfaces (and, more generally, CR manifolds of hypersurface type) which have *degenerate* Levi forms. Before describing our main results, we should mention that another approach to the biholomorphic equivalence problem is via normal forms. Normal forms for certain types of Levi degeneracies were studied by the author in [E3–4]; another class of Levi degeneracies in $\mathbb{C}^2$ was considered by Wong [Wo]. However, at least to the best of the author’s knowledge, the geometric approach as described above has not been previously studied for CR manifolds with degenerate Levi forms, except in the case where the manifold is locally biholomorphically equivalent to one of the form $\tilde{M} \times \mathbb{C}^p$ with $\tilde{M}$ Levi nondegenerate. The latter situation was studied by M. Freeman [F2]. The situation in the present paper is quite the opposite; the manifolds considered here are not locally equivalent to $\tilde{M} \times \mathbb{C}^p$ for any manifold $\tilde{M}$ and $p > 0$.

The idea in this paper is to use the cubic form as a complement to the degenerate Levi form. We shall restrict our attention to 5-dimensional manifolds. As the number of dimensions increase the algebraic complexity of the resulting tensors and the number of cases that need to be distinguished greatly increase.

The paper is organized as follows. Our main results, and some applications including a unique continuation principle, are explained in §0.2. §1 is devoted to preliminary material including basic definitions and properties of Levi uniform CR manifolds. The necessary constructions for the main results are given in §2–3, and in §4 a discussion and characterization of the tube over the light cone is given. A proof of the unique continuation principle is given in §5. In §6, some examples of everywhere Levi degenerate hypersurfaces that arise e.g. in PDE theory are given.

Acknowledgement. The author would like to thank the anonymous referee for many comments that helped to improve the presentation of this work.

0.2. The main results. Our main results concern real hypersurfaces M in $\mathbb{C}^3$ (or, more generally, 5-dimensional CR manifolds of hypersurface type), which are uniformly Levi degenerate in the sense that the Levi form has one nonzero and one zero eigenvalue in a neighborhood of a distinguished point $p_0 \in M$ (*Levi uniform* of rank 1 according to Definition 1.5 below). Such hypersurfaces $M \subset \mathbb{C}^3$ are foliated by complex curves (cf. Proposition 1.12), but in general this foliation cannot be (even locally) biholomorphically straightened; i.e. M is not locally equivalent to $\tilde{M} \times \mathbb{C}$ for any hypersurface $\tilde{M} \subset \mathbb{C}^2$. We should point out that obstructions to a local straightening of a complex foliation of a CR manifold was investigated by Freeman [F2]. In the present paper, we shall require M to be 2-nondegenerate at p_0 (see section 1 or [BER, Chapter XI]). The latter condition guarantees that M is *holomorphically nondegenerate* (see Stanton [Sta1–2]; cf. also [BER, Chapter XI]) and, in particular, the foliation of M cannot be locally biholomorphically straightened. In fact, if $M \subset \mathbb{C}^3$ is real-analytic, connected, and everywhere Levi degenerate, then there is a proper real-analytic subvariety $V \subset M$ such that either M is locally equivalent to $\tilde{M} \times \mathbb{C}$, for $\tilde{M} \subset \mathbb{C}^2$, near each $p \in M \setminus V$ (locally biholomorphically straight) or M is Levi uniform of rank 1 and 2-nondegenerate at every $p \in M \setminus V$.

The most important example (indeed, the “standard model”) of a hypersurface of the type considered here is the tube in $\mathbb{C}^3$ over the light cone in $\mathbb{R}^3$.

Example 0.2.1. The tube $\Gamma_{\mathbb{C}}$ in $\mathbb{C}^3$ over the light cone in $\mathbb{R}^3$, i.e. the variety defined by

$$(0.2.2) \quad (\operatorname{Re} Z_1)^2 + (\operatorname{Re} Z_2)^2 - (\operatorname{Re} Z_3)^2 = 0,$$

is Levi uniform of rank 1 at every nonsingular point, i.e. at every point where it is a real submanifold. In particular, $\Gamma_{\mathbb{C}}$ is Levi degenerate at every point, but it cannot be locally biholomorphically straightened. The latter fact was first proved by Sergeev–Vladimirov (see e.g. [SV]) by calculating the Freeman obstructions mentioned above. The reader can also verify that $\Gamma_{\mathbb{C}}$ is 2-nondegenerate at every nonsingular point. This example will be discussed in greater detail in §4 below. (See also [E3], where $\Gamma_{\mathbb{C}}$ is further discussed in connection with a normal form for 2-nondegenerate hypersurfaces in $\mathbb{C}^3$.)

Let us also point out that the biholomorphically invariant geometry of the tube over the light cone in $\mathbb{C}^4$ plays an important role in e.g. axiomatic quantum field theory since $\Gamma_{\mathbb{C}}$ bounds the so-called past and future tubes; see e.g. Sergeev–Vladimirov

[SV]. Other examples of everywhere Levi degenerate hypersurfaces that arise naturally are given, for motivation, in section 6.

For the class of hypersurfaces described above, we define a new CR invariant $\hat{k}$ (see (3.8)). One of our main results is the following. We refer the reader to section 1 for relevant definitions.

Theorem 1. *Let M be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0 \in M$.*

- (a) *If $\hat{k}(p_0) \neq 0$, then there exist an open neighborhood M_0 of p_0 in M , a principal fiber bundle $P_0 \rightarrow M_0$ with a 2-dimensional group $G_0 \subset GL(\mathbb{R}^3)$, and a 1-form $\underline{\omega}^0$ on P_0 which defines an isomorphism between $T_u P_0$ and $\mathbb{R}^7$ for every $u \in P_0$ and reduces the CR structure on M_0 to a parallelism.*
- (b) *If $|\hat{k}(p_0)| \neq 2$, then there exist an open neighborhood M_1 of p_0 in M , a principal fiber bundle $P_1 \rightarrow M_1$ with a 1-dimensional group $G_1 \subset GL(\mathbb{R}^3)$, and a 1-form $\underline{\omega}^1$ on P_1 which defines an isomorphism between $T_u P_1$ and $\mathbb{R}^6$ for every $u \in P_1$ and reduces the CR structure on M_1 to a parallelism.*

Theorem 1 will be a consequence of the more detailed Theorems 3.1.37 and 3.2.9.

The reader should observe that if $\hat{k}(p_0) \neq 0$ and $|\hat{k}(p_0)| \neq 2$, then there are two different bundles over a neighborhood of p_0 . Both P_0 and P_1 are submanifolds of a larger principal G -bundle $P \rightarrow M$, which is defined in a more general context in section 2. In general, P_1 , although of lower dimension, is not a submanifold of P_0 . The groups G_0 and G_1 are subgroups of G . The bundle P_1 is a reduction of P , whereas P_0 is in general not; the action of G_0 on P_0 differs slightly from the action of G_0 on P . The bundles P_0, P_1 are uniquely defined by intrinsic conditions on M . Hence, P_0 extends as a bundle over the (open) subset of points on M where $\hat{k} \neq 0$, and P_1 as a bundle over the subset of points where $|\hat{k}| \neq 2$.

We should also mention that the groups G_0, G_1 , and hence the bundles P_0, P_1 , in Theorem 1 are disconnected and have two components. In order to obtain a connected bundle, we have to choose an orientation for the CR structure at p_0 as explained in §3 (see Theorems 3.1.37 and 3.2.9).

As mentioned above, the most important example is the tube $\Gamma_{\mathbb{C}}$ over the light cone for which the invariant $\hat{k} \equiv 2i$. We shall now characterize $\Gamma_{\mathbb{C}}$ among all M , as in Theorem 1, by a curvature condition in the spirit of the characterization of the sphere among strongly pseudoconvex hypersurfaces as described section 0.1. There is a subgroup H of $GL(\mathbb{C}^4)$ and a subgroup H_0 of H such that H can be viewed as a principal fiber bundle over $\Gamma_{\mathbb{C}}$ with group H_0 . The matrix valued Maurer-Cartan forms Π of H define a Cartan connection on $\Gamma_{\mathbb{C}}$ valued in $\mathfrak{h}$, the Lie algebra of H , with vanishing curvature $\Omega = d\Pi - \Pi \wedge \Pi$. (All this is explained in detail in section 4.) Now, since $\hat{k}$ is an invariant which is $\equiv 2i$ for $\Gamma_{\mathbb{C}}$, a necessary condition for a CR manifold M , as in Theorem 1, to be locally equivalent to $\Gamma_{\mathbb{C}}$ is $\hat{k} \equiv 2i$ for M . For such M , we can identify the group G_0 with the group H_0 , and construct, using $\underline{\omega}$, a $\mathfrak{h}$ -valued 1-form Π which, unfortunately, in general is not a Cartan connection. However, we have the following result, which is a consequence of the more detailed Theorem 4.31.

Theorem 2. *Let M a real-analytic CR manifold satisfying the conditions in Theorem 1 with $\hat{k} \equiv 2i$ near $p_0 \in M$. Then there exists a $\mathfrak{h}$ -valued 1-form Π on the principal bundle $P_0 \rightarrow M_0$, where $M_0 \subset M$ and P_0 are given by Theorem 1, which*

gives an isomorphism between $T_u P_0$ and $\mathfrak{h}$ for every $u \in P_0$ and with the following property. There exists a real-analytic CR diffeomorphism $f: M \rightarrow \Gamma_{\mathbb{C}}$, defined near $p_0 \in M$, if and only if the curvature

$$(0.2.3) \quad \Omega := d\Pi - \Pi \wedge \Pi$$

vanishes identically near p_0 .

Remark. Examples of real-analytic hypersurfaces which satisfy the conditions in Theorem 2 at generic points include tubes over homogeneous algebraic varieties in $\mathbb{R}^3$ (Proposition 6.13) and tubes over surfaces in $\mathbb{R}^3$ which are everywhere characteristic for the wave operator (Proposition 6.27). An example of a real-analytic hypersurface satisfying the conditions in the theorem (at generic points) for which the curvature $\Omega \not\equiv 0$ is given by Freeman's hypersurface $\{Z \in \mathbb{C}^3: (\operatorname{Re} Z_1)^3 + (\operatorname{Re} Z_2)^3 - (\operatorname{Re} Z_3)^3 + 0\}$. This fact is a consequence of Proposition 6.36.

We conclude this section by giving two applications of Theorem 1. We shall use the notation $\operatorname{Aut}(M, p_0)$ for the stability group of a CR manifold M at $p_0 \in M$, i.e. the group of germs at p_0 of local smooth CR diffeomorphisms $f: (M, p_0) \rightarrow (M, p_0)$. Suppose that M satisfies the conditions of Theorem 1 at p_0 . Pick any point $u \in P_{p_0}$, where $P \rightarrow M$ is the principal G_0 bundle given by Theorem 1 and P_p denotes the fiber over $p \in M$. By [K, Theorem 3.2] and Theorem 1, the group $\operatorname{Aut}(M, p_0)$ embeds as a closed submanifold of the fiber $P_{p_0} \cong G_0$ via the mapping

$$(0.2.4) \quad \operatorname{Aut}(M, p_0) \ni f \mapsto F(u) \in P_{p_0},$$

where $F: P \rightarrow P$ is the lift of f as in the diagram (0.1.1). Thus, $\dim \operatorname{Aut}(M, p_0)$ is at most 2 if the invariant $|\hat{k}(p_0)| = 2$ and at most 1 if $|\hat{k}(p_0)| \neq 2$. We formulate this as follows.

Corollary 3. *Let M be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0 \in M$. Then $\dim \operatorname{Aut}(M, p_0) \leq 2$.*

The bound in Corollary 3 cannot be improved, since $\dim \operatorname{Aut}(\Gamma_{\mathbb{C}}, p) = 2$ for any $p \in \Gamma_{\mathbb{C}}$ as is shown in section 4. We should mention that in the recent preprint [Er], it was shown that the bound $\dim \operatorname{Aut}(M, p_0) \leq 3$ holds for the class of all real-analytic 2-nondegenerate hypersurfaces in $\mathbb{C}^3$; observe that when M is real-analytic then it follows from the reflection principle in [BJT] that every $f \in \operatorname{Aut}(M, p_0)$ is real-analytic.

As our final application, we give a strong unique continuation principle for CR mappings. We shall use the notation $f: (M, p_0) \rightarrow (M', p'_0)$ to denote the fact that f is a smooth mapping defined in a neighborhood of p_0 in M , sending this neighborhood into M' and $f(p_0) = p'_0$.

Theorem 4. *Let M, M' be 5-dimensional CR manifolds of hypersurface type, and assume that both M and M' are Levi uniform and 2-nondegenerate at $p_0 \in M$ and $p'_0 \in M'$, respectively. Let $f^1, f^2: (M, p_0) \rightarrow (M', p'_0)$ be smooth CR diffeomorphisms. If $j_{p_0}^1 f^0 = j_{p_0}^1 f^1$, where $j_p^1 f$ denotes the first order jet at $p \in M$ of a mapping $f: M \rightarrow M'$, then $f^1 \equiv f^2$ near p_0 .*

A similar unique continuation principle follows from the work of Chern and Moser [CM] for Levi nondegenerate CR manifolds. We should point out that Theorem

4 is new even for embedded real hypersurfaces. In addition, for embedded real hypersurfaces one can relax the condition that the mappings are diffeomorphisms. To state a more general result in this case, we need a definition. If M is minimal at p_0 (see [BER, Chapter I] for the definition; we mention here only the fact that if a hypersurface M is 2-nondegenerate, then it is minimal), then any smooth CR mapping $f: (M, p_0) \rightarrow (M', p'_0)$ attains its maximal rank on a dense open subset of any sufficiently small neighborhood of p_0 (see [BR3]). Following [BR3], we shall call this integer the *generic rank* of f near p_0 .

Theorem 5. *Let $M, M' \subset \mathbb{C}^3$ be smooth real hypersurfaces and assume that both M and M' are Levi uniform and 2-nondegenerate at $p_0 \in M$ and $p'_0 \in M'$, respectively. Let $f^0: (M, p_0) \rightarrow (M', p'_0)$ be a smooth CR mapping such that its generic rank near p_0 is at least 3. If $f^1: (M, p_0) \rightarrow (M', p'_0)$ is a smooth CR mapping with $j_{p_0}^1 f^0 = j_{p_0}^1 f^1$, then $f^0 \equiv f^1$ near p_0 . Moreover, f^0 is a local CR diffeomorphism near p_0 .*

Theorems 4 and 5 will be proved in section 5 below.

Remarks. (i) An inspection of the proofs of Theorems 4 and 5 shows that it is not necessary for the mappings to be C^∞ for the conclusion to hold. Indeed, it suffices to assume that the mappings are C^2 . On the other hand, even if we assume only C^2 -regularity of the mappings, the proof (in combination with standard result on regularity for solutions of PDE) actually yields C^∞ -smoothness. Moreover, if M and M' in addition are assumed to be real-analytic, then the conclusion is that the mappings are real-analytic. (cf. also [Han], [Hay].) Hence, the method of proof also gives regularity results, and a reflection principle in the real-analytic case. The reflection principle, however, follows from the more general one in [BJT] (as mentioned above) once it is shown that the mappings are local diffeomorphisms (i.e. the last conclusion in Theorem 5). Since this part of Theorem 5 essentially follows from previous results due to Baouendi and Rothschild (see the proof), this reflection principle should be regarded as a known result.

(ii) The generic rank condition imposed on f^0 in Theorem 5 is necessary for the conclusion to hold. In the example in §5.3 below, we show that there are real hypersurfaces $M, M' \subset \mathbb{C}^3$ with points $p_0 \in M$ and $p'_0 \in M'$ as in Theorem 5, and a smooth CR mapping $f^0: (M, p_0) \rightarrow (M', p'_0)$ with generic rank 2 near p_0 such that $j_{p_0}^k(f^0)$ equals the k -jet of the constant mapping for any nonnegative integer k , but f^0 is not constant in any neighborhood of p_0 .

1. PRELIMINARIES

Let M be a CR manifold with CR bundle $\mathcal{V}$. Recall that this means that $\mathcal{V}$ is a subbundle of the complexified tangent bundle $\mathbb{C}TM$ such that $\mathcal{V}_p \cap \bar{\mathcal{V}}_p = \{0\}$ for every $p \in M$, and $\mathcal{V}$ is *formally integrable* i.e. any commutator between sections of $\mathcal{V}$ is again a section of $\mathcal{V}$; sections of $\mathcal{V}$ will henceforth be called *CR vector fields*. We shall denote the CR dimension of M , i.e. the (complex) dimension of the fibers $\mathcal{V}_p$ for $p \in M$, by n . We shall assume that M is of *hypersurface type*, i.e. the complex dimension of $T_p^0 M := (\mathcal{V}_p \oplus \bar{\mathcal{V}}_p)^\perp \subset \mathbb{C}T_p^* M$, for $p \in M$, is one. In particular, the dimension of M is $2n + 1$. For the remainder of this paper unless explicitly stated otherwise, all CR manifolds will be of hypersurface type. The bundle $T^0 M \subset \mathbb{C}T^* M$ is called the *characteristic bundle*, and real sections of $T^0 M$ are called *characteristic forms*. The subbundle $T' M \subset \mathbb{C}T^* M$, defined at $p \in M$ by

$T'_p M := \mathcal{V}_p^\perp$, is called the *holomorphic cotangent bundle*. The formal integrability of $\mathcal{V}$ is equivalent to the following property of $T'_p M$: If ω is a section of $T'_p M$, then $d\omega$ is a section of the ideal generated by $T'_p M$ in the exterior algebra of $\mathbb{C}TM$.

Let $L_{\bar{1}}, \dots, L_{\bar{n}}$ be a basis for the CR vector fields near some distinguished point $p_0 \in M$. Also, let θ be a nonvanishing characteristic form near the same point p_0 . We shall define a linear operator $\mathcal{T}_{\bar{A}}$ on the holomorphic 1-forms on M , i.e. the sections of $T'_p M$, near p_0 by

$$(1.1) \quad \mathcal{T}_{\bar{A}}\omega := \frac{1}{2i} L_{\bar{A}} \lrcorner d\omega,$$

where $\lrcorner$ denotes the usual contraction by a vector field. Note that, for a holomorphic 1-form ω , $\mathcal{T}_{\bar{A}}\omega$ is again a holomorphic 1-form which can be thought of as the Lie derivative of ω along the vector field $L_{\bar{A}}$.

For $p \in M$ near p_0 and positive integers k , we define the subspace $E_{k,p} \subset T'_p M$ as the (complex) linear span of θ_p and $(\mathcal{T}_{\bar{A}_j} \dots \mathcal{T}_{\bar{A}_1} \theta)_p$, for all $1 \leq j \leq k$ and all j -tuples $(A_1, \dots, A_j) \in \{1, \dots, n\}^j$. We define $E_{0,p}$ to be $T'_p M$. The CR manifold M is said to be *finitely nondegenerate* at p of

$$(1.2) \quad E_{k,0} = T'_p M$$

for some integer $k \geq 1$, and k_0 -nondegenerate at p if k_0 is the smallest integer k for which (1.2) holds. It was shown in [E2] (see also [BER, Chapter XI]) that this definition is consistent with the one for real hypersurfaces of $\mathbb{C}^{n+1}$ given in [BHR]. (These notions can also be extended to CR manifolds of arbitrary codimension; see e.g. [BER].)

The *Levi form*, $\mathcal{L}_\theta$, of M at p and θ_p is a Hermitian form on $\mathcal{V}_p$ which, relative to the basis $L_{\bar{1},p}, \dots, L_{\bar{n},p}$, is represented by a Hermitian $n \times n$ matrix $(g_{\bar{A}B}(p))_{1 \leq A,B \leq n}$, where

$$(1.3) \quad g_{\bar{A}B}(p) := \frac{1}{2i} \langle \theta_p, [L_B, L_{\bar{A}}]_p \rangle.$$

We use here, and throughout this paper, the convention that $L_B = \overline{L_{\bar{B}}}$. The Levi form can also be expressed using the Lie derivative as follows

$$(1.4) \quad g_{\bar{A}B}(p) = \langle (T_{\bar{A}}\theta)_p, L_{B,p} \rangle.$$

From this observation, we see that 1-nondegeneracy of M at p is equivalent to the classical notion of *Levi nondegeneracy* of M at p .

Definition 1.5. A CR manifold M of hypersurface type is *1-uniform* or *Levi uniform* (of rank r) at $p \in M$ if the rank of the Levi form is constant (and equal to r) in a neighborhood of p .

Observe that the extremal cases of CR manifolds which are Levi uniform of rank 0 or n at a point p are precisely those which are Levi flat or Levi nondegenerate, respectively, at p . Such CR manifolds which, in addition, are real-analytic are by now fairly well understood: a real-analytic Levi flat CR manifold is locally CR equivalent to the real hyperplane $\text{Im } Z_{n+1} = 0$ in $\mathbb{C}^{n+1}$, and a theory for real-analytic Levi nondegenerate CR manifolds was developed by E. Cartan [C1-2], Tanaka [T1-2], and Chern–Moser [CM].

The reader should also observe that any real-analytic CR manifold is Levi uniform outside a proper real-analytic subvariety (in particular, on a dense open subset). Let us point out that the tube over the light cone (see Example 0.2.1) is an example of a Levi uniform CR manifold which is neither Levi nondegenerate nor Levi flat.

We shall denote by $\mathfrak{N}_p \subset \bar{\mathcal{V}}_p$ the *Levi nullspace* at p , i.e. those vectors $X_p \in \bar{\mathcal{V}}_p$ for which the linear form $Y_p \mapsto \mathcal{L}_\theta(Y_p, \bar{X}_p)$ on $\mathcal{V}_p$ is zero. If M is Levi uniform (of rank r) at $p_0 \in M$, then the subspaces $\mathfrak{N}_p$ for p near p_0 form a (rank $n - r$) subbundle $\mathfrak{N}$ of $\bar{\mathcal{V}}$. From now on, we assume that M is Levi uniform of rank r , with $0 < r < n$, in a neighborhood of p_0 (to which we restrict our attention). We may arrange our basis for the CR vector fields $L_{\bar{1}}, \dots, L_{\bar{n}}$ so that $L_{r+1}, \dots, L_n$ is a basis for the sections of $\mathfrak{N}$ near that point. The Levi form, for p near p_0 , then takes the form

$$(1.6) \quad (g_{\bar{A}B}) = \begin{pmatrix} g_{\bar{\alpha}\beta} & 0 \\ 0 & 0 \end{pmatrix},$$

where $(g_{\bar{\alpha}\beta})_{1 \leq \alpha, \beta \leq r}$ is an $r \times r$ nondegenerate Hermitian matrix of smooth functions. In what follows, we shall use the summation convention and also the convention that capital roman indices $A, B, \dots$ run over the integers $\{1, \dots, n\}$ and Greek indices $\alpha, \beta, \dots$ run over $\{1, \dots, r\}$. The *cubic tensor* (see e.g. Freeman [F2] and Webster [We2]) can now be represented near p_0 by $n - r$ $n \times n$ matrices $(h_{\bar{A}\bar{B}k})_{1 \leq A, B \leq n, k = r+1, \dots, n}$, of smooth functions, where

$$(1.7) \quad h_{\bar{A}\bar{B}k} := \langle \mathcal{T}_{\bar{B}} \mathcal{T}_{\bar{A}} \theta, L_k \rangle.$$

The cubic tensor is usually defined using commutators of three vector fields (see [We2]) but for a Levi uniform manifold it is not difficult to see (cf. the computation in the proof of Proposition 1.8 below) that the commutator definition coincides with the one using Lie derivatives given here. We should point out that higher order tensors, or Levi forms, have also been studied in the literature (see e.g. [F2], [We2], [E4]). We shall not discuss these further in this work.

Proposition 1.8. *Assume that M is Levi uniform of rank r at p_0 . Then, in the notation introduced above, for every $k = r + 1, \dots, n$ and all p in a neighborhood of p_0 , it holds that $h_{\bar{A}\bar{B}k} = 0$ whenever A or B belongs to $\{r + 1, \dots, n\}$.*

Proof. Using a well known identity (see e.g. [He, Chapter I.2]; see also the remark concerning our normalization of the pairing $\langle \cdot, \cdot \rangle$ in [E4]), we have

$$(1.9) \quad \begin{aligned} h_{\bar{A}\bar{B}k} &:= \langle \mathcal{T}_{\bar{B}} \mathcal{T}_{\bar{A}} \theta, L_k \rangle = \langle d(\mathcal{T}_{\bar{A}} \theta), L_{\bar{B}} \wedge L_k \rangle \\ &= L_{\bar{B}}(\langle \mathcal{T}_{\bar{A}} \theta, L_k \rangle) - L_k(\langle \mathcal{T}_{\bar{A}} \theta, L_{\bar{B}} \rangle) - \langle \mathcal{T}_{\bar{A}} \theta, [L_{\bar{B}}, L_k] \rangle \\ &= -\langle \mathcal{T}_{\bar{A}} \theta, [L_{\bar{B}}, L_k] \rangle, \end{aligned}$$

where the last identity follows from the facts that $\langle \mathcal{T}_{\bar{A}} \theta, L_k \rangle \equiv 0$ and $\langle \mathcal{T}_{\bar{A}} \theta, L_{\bar{B}} \rangle \equiv 0$. It is not difficult to see that the matrices $h_{\bar{A}\bar{B}k}$ are symmetric (cf. [E4]), so to prove the proposition it suffices to show $h_{\bar{A}\bar{l}k} = 0$ in a neighborhood of p_0 for $k, l \geq r + 1$, i.e. $\langle \mathcal{T}_{\bar{A}} \theta, [L_{\bar{l}}, L_k] \rangle = 0$ in view of (1.9). To this end note, using the fact that $L_{k,p}$ and $L_{l,p}$ are null vectors for the Levi form at every p near p_0 , that

$$(1.13) \quad \langle \mathcal{T}_{\bar{A}} \theta, [L_{\bar{l}}, L_k] \rangle = -\langle \theta, [L_{\bar{A}}, [L_{\bar{l}}, L_k]] \rangle.$$

Thus, by also using the Jacobi identity, we obtain

$$(1.11) \quad \langle \mathcal{T}_{\bar{A}}\theta, [L_{\bar{l}}, L_k] \rangle = \langle \theta, [L_{\bar{l}}, [L_k, L_{\bar{A}}]] \rangle + \langle \theta, [L_k, [L_{\bar{A}}, L_{\bar{l}}]] \rangle.$$

The second term on the right hand side of (1.11) vanishes since $[L_{\bar{A}}, L_{\bar{l}}]$ is a CR vector field by the formal integrability of $\mathcal{V}$ and L_k is a null vector field for the Levi form. To show that the first term also vanishes, we must show that $[L_k, L_{\bar{A}}]$ is a section of $\mathcal{V} \oplus \bar{\mathcal{V}}$. This fact follows again from the fact that L_k is a null vector field for the Levi form since the latter is equivalent to $\langle \theta, [L_k, L_{\bar{A}}] \rangle = 0$ for every $A = 1, \dots, n$. The proof of Proposition 1.8 is complete. $\square$

Let us digress briefly to note the following result which, although of no importance for the remainder of this paper, follows from (the proof of) Proposition 1.8 above. The result is most certainly known, but we include the brief proof for the convenience of the reader.

Proposition 1.12. *If $M \subset \mathbb{C}^{n+1}$ is a real hypersurface which is Levi uniform of rank $r < n$ at $p_0 \in M$, then M is foliated by complex manifolds of dimension $n - r$ in a neighborhood of p_0 .*

Remark 1.13. As mentioned in the introduction, the foliation given by Proposition 1.12, even when M is real-analytic, can in general not be locally straightened, i.e. it is in general not true that M , as in the proposition, is CR equivalent to a real hypersurface of the form $\tilde{M} \times \mathbb{C}^r \subset \mathbb{C}^{n+1}$, where $\tilde{M}$ is a real hypersurface in $\mathbb{C}^{n+1-r}$ and $r > 0$. Indeed, if M is CR equivalent to $\tilde{M} \times \mathbb{C}^r$ (which is a *holomorphically degenerate* hypersurface), then it cannot be finitely nondegenerate at any point (see e.g. [BER, Chapter XI]). Obstructions to local straightening were also studied by Freeman [F2].

Proof of Proposition 1.12. An immediate consequence of Proposition 1.8 and (1.9) is that, for $k, l \geq r + 1$,

$$(1.14) \quad [L_k, L_{\bar{l}}] = \sum_{m=r+1}^n (a_{k\bar{l}}^m L_m + b_{k\bar{l}}^{\bar{m}} L_{\bar{m}}),$$

where the $a_{k\bar{l}}^m$ and $b_{k\bar{l}}^{\bar{m}}$ are smooth functions satisfying $a_{k\bar{l}}^m + \overline{b_{k\bar{l}}^{\bar{m}}} = 0$. A similar argument shows that $\mathfrak{N}$ is formally integrable, i.e. the commutator $[L_k, L_l]$, for $k, l \geq r + 1$, is also a section of $\mathfrak{N}$. Thus, by the Frobenius theorem, M is foliated near p_0 by $2(n - r)$ -dimensional integral manifolds of $\text{Re } L_k, \text{Im } L_k, k = r + 1, \dots, n$. By the Newlander–Nirenberg theorem, these manifolds are $(n - r)$ -dimensional complex submanifolds in $\mathbb{C}^{n+1}$. $\square$

Returning to the cubic tensor, we observe that Proposition 1.8 shows that the matrices $(h_{\bar{A}\bar{B}k})$ representing it are of the form

$$(1.15) \quad (h_{\bar{A}\bar{B}k}) = \begin{pmatrix} h_{\bar{\alpha}\bar{\beta}k} & 0 \\ 0 & 0 \end{pmatrix}.$$

We conclude this section with the following observation, whose proof is immediate and left to the reader, characterizing 2-nondegeneracy for Levi uniform CR manifolds of rank r in terms of the $r \times r$ matrices $(h_{\bar{\alpha}\bar{\beta}k})$ in (1.15).

Proposition 1.16. *Assume that M is Levi uniform of rank r at p_0 . Then, in the notation introduced above, M is 2-nondegenerate at p_0 if and only if the symmetric matrices $(h_{\bar{\alpha}\bar{\beta}k}(p_0))_{1 \leq \alpha, \beta \leq r}, k = r + 1, \dots, n$, are linearly independent over $\mathbb{C}$.*

2. A G -STRUCTURE FOR LEVI UNIFORM CR MANIFOLDS OF RANK $n - 1$

We keep the notation from the previous section. However, from now on, we shall restrict ourselves to the case $r = n - 1$, i.e. the case where the rank of the Levi form near p_0 is $n - 1$. Thus, we assume that M is a smooth CR manifold (of hypersurface type) of CR dimension n which is Levi uniform of rank $n - 1$ at a distinguished point $p_0 \in M$. We shall restrict our attention to a small neighborhood of p_0 . In what follows, M will denote a sufficiently small neighborhood of p_0 . We have the following invariant subbundles of the cotangent bundle $\mathbb{C}T^*M$

$$(2.1) \quad T^0M \subset T''M \subset T'M,$$

where T^0M and $T'M$ were introduced in §1 and where $T''M$ is defined by

$$(2.2) \quad T''M = \{\omega_p \in T'_pM : \langle \omega_p, L_p \rangle = 0, \forall L_p \in \mathfrak{N}_p\}.$$

Observe that

$$(2.3) \quad \dim T_p^0M = 1, \dim T_p''M = n, \dim T_p'M = n + 1.$$

Let $\theta, \theta^1, \dots, \theta^n$ be a basis for the holomorphic 1-forms (i.e. sections of $T'M$) with the additional properties that θ is real and a basis for the sections of T^0M and $\theta, \theta^1, \dots, \theta^{n-1}$ is a basis for the sections of $T''M$. Any other such basis $\tilde{\theta}, \tilde{\theta}^1, \dots, \tilde{\theta}^n$ is related to $\theta, \theta^1, \dots, \theta^n$ by

$$(2.4) \quad \begin{pmatrix} \tilde{\theta} \\ \tilde{\theta}^\alpha \\ \tilde{\theta}^n \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ u^\alpha & u_\beta^\alpha & 0 \\ \xi^\alpha & \xi_\beta^\alpha & \xi \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\beta \\ \theta^n \end{pmatrix},$$

where the coefficients in the $(n + 1) \times (n + 1)$ matrix in (2.4) are smooth functions (all complex valued, except u which is real valued); also, recall that we are using the summation convention, and Greek indices run over the set $\{1, \dots, n - 1\}$ since $r = 1$ here.

Observe that $T_p''M \cap \overline{T_p''M} = T_p^0M$ and $T''M \cup \overline{T''M}$ is a rank $2n - 1$ subbundle of $\mathbb{C}T^*M$. The 1-forms $\theta, \theta^\alpha, \theta^{\bar{\alpha}}$, where $\theta^{\bar{\alpha}} = \overline{\theta^\alpha}$ as mentioned in §1, yield a coframe for bundle $T''M \cup \overline{T''M}$. Consider the bundle $Y \rightarrow M$ consisting of all such coframes $(\omega, \omega^\alpha, \omega^{\bar{\alpha}})^\tau$,

$$(2.5) \quad \begin{pmatrix} \omega \\ \omega^\alpha \\ \omega^{\bar{\alpha}} \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ u^\alpha & u_\beta^\alpha & 0 \\ \overline{u^\alpha} & 0 & \overline{u_\beta^\alpha} \end{pmatrix} \begin{pmatrix} \theta \\ \theta^\beta \\ \theta^{\bar{\beta}} \end{pmatrix},$$

where $(u, u^\alpha, u_\beta^\alpha) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{C}^{n-1} \times GL(\mathbb{C}^{n-1})$. If we let $G \subset GL(\mathbb{C}^{2n-1})$ denote the group consisting of matrices of the form

$$(2.6) \quad S = \begin{pmatrix} u & 0 & 0 \\ u^\alpha & u_\beta^\alpha & 0 \\ \overline{u^\alpha} & 0 & \overline{u_\beta^\alpha} \end{pmatrix}, \quad (u, u^\alpha, u_\beta^\alpha) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{C}^{n-1} \times GL(\mathbb{C}^{n-1}),$$

then $Y \rightarrow M$ is a principal fiber bundle over M with group G ; (2.5) gives a trivialization of Y in which $(u, u^\alpha, u_\beta^\alpha)$, or S given by (2.6), are (global) coordinates of Y . We denote by $\mathfrak{g}$ the Lie algebra of G , i.e. the space of matrices

$$(2.7) \quad T = \begin{pmatrix} v & 0 & 0 \\ v^\alpha & v_\beta^\alpha & 0 \\ \overline{v^\alpha} & 0 & \overline{v_\beta^\alpha} \end{pmatrix}, \quad (v, v^\alpha, v_\beta^\alpha) \in \mathbb{R} \times \mathbb{C}^{n-1} \times \mathcal{M}(\mathbb{C}^{n-1}),$$

where $\mathcal{M}(\mathbb{C}^{n-1})$ denotes the space of all $(n-1) \times (n-1)$ matrices. If we pull back the forms $\theta, \theta^A, \theta^{\bar{A}}$ (where capital Roman indices run over $\{1, \dots, n\}$) to Y , still denoting the pulled-back forms by $\theta, \theta^A, \theta^{\bar{A}}$, then (2.5) defines 1-forms $\omega, \omega^\alpha, \omega^{\bar{\alpha}}$ on Y . The reader can verify that the latter 1-forms are invariantly defined on Y , i.e. independent of the initial choice of $\theta, \theta^\alpha, \theta^{\bar{\alpha}}$ above.

Differentiating the 1-forms $\omega, \omega^\alpha, \omega^{\bar{\alpha}}$ we obtain

$$(2.8) \quad d \begin{pmatrix} \omega \\ \omega^\alpha \\ \omega^{\bar{\alpha}} \end{pmatrix} = dSS^{-1} \wedge \begin{pmatrix} \omega \\ \omega^\beta \\ \omega^{\bar{\beta}} \end{pmatrix} + Sd \begin{pmatrix} \theta \\ \theta^\beta \\ \theta^{\bar{\beta}} \end{pmatrix},$$

where $S \in G$ is given by (2.6). The elements of the matrix valued 1-form dSS^{-1} are Maurer–Cartan forms for the Lie group G (see e.g. [G]). Let $L, L_A, L_{\bar{A}}$ denote a dual basis relative to $\theta, \theta^A, \theta^{\bar{A}}$. Thus, the $L_{\bar{A}}$ form a basis for the CR vector fields, $L_A = \overline{L_{\bar{A}}}$, and L_n spans the Levi nullbundle $\mathfrak{N}$. We shall use the notation and representations introduced in §1 for the Levi form and cubic tensor (relative to the bases chosen). Since M is Levi uniform of rank $n-1$ near p_0 , the Levi form $(g_{\bar{A}B})$ satisfies (1.6) with $r = n-1$, and the cubic tensor $(h_{\bar{A}\bar{B}n})$ satisfies, by Proposition 1.8, $h_{\bar{n}\bar{B}n} = h_{\bar{A}\bar{n}n} = 0$ in a neighborhood of p_0 . Moreover, the matrix $(h_{\bar{\alpha}\bar{\beta}n})$ is symmetric. $(g_{\bar{\alpha}\beta})$ is an invertible Hermitian matrix and we shall denote its inverse by $(g^{\alpha\beta})$, i.e. $g_{\bar{\alpha}\beta}g^{\bar{\alpha}\gamma} = \delta_\beta^\gamma$. Using the formal integrability of $\mathcal{V}$ and the fact that θ is real, we obtain

$$(2.9) \quad d\theta = ig_{\bar{\alpha}\beta}\theta^{\bar{\alpha}} \wedge \theta^\beta + \phi \wedge \theta,$$

where ϕ is a real 1-form. Similarly by the formal integrability, we can write

$$(2.11) \quad d\theta^\alpha = \eta^\alpha \wedge \theta + \eta_\beta^\alpha \wedge \theta^\beta + h_{\bar{B}}^\alpha \theta^{\bar{B}} \wedge \theta^n,$$

for some 1-forms η^α and η_β^α , and some matrix $(h_{\bar{B}}^\alpha)$.

Lemma 2.12. *In the notation introduced above, we have*

$$(2.13) \quad h_n^\beta = 0, \quad 2ig^{\bar{\alpha}\beta}h_{\bar{\alpha}\bar{\gamma}n} = h_{\bar{\gamma}}^\beta.$$

Proof. Recall the operators $\mathcal{T}_{\bar{A}}$ introduced in §1. Observe that, by the definition of $T''M$ and the fact that L_n spans $\mathfrak{N}$, the 1-forms θ and $\mathcal{T}_{\bar{\alpha}}\theta$ form a basis for the sections of $T''M$, and $\mathcal{T}_{\bar{n}}\theta = c\theta$ for some smooth function c . Indeed, we have

$$(2.14) \quad \theta^\beta = g^{\bar{\alpha}\beta}\mathcal{T}_{\bar{\alpha}}\theta + c^\beta\theta,$$

for some smooth functions c^β , as is straightforward to verify. In view of (2.14), a direct calculation using the definition of the cubic tensor shows

$$(2.15) \quad \langle L_{\bar{B}} \lrcorner d\theta^\beta, L_n \rangle = 2ig^{\bar{\alpha}\beta} h_{\bar{\alpha}\bar{B}n}.$$

On the other hand, a calculation using (2.11) shows

$$(2.16) \quad \langle L_{\bar{B}} \lrcorner d\theta^\beta, L_n \rangle = h_{\bar{B}}^\beta,$$

which completes the proof. $\square$

Thus, we rewrite (2.11) as follows

$$(2.17) \quad d\theta^\alpha = \eta^\alpha \wedge \theta + \eta_\beta^\alpha \wedge \theta^\beta + h_{\bar{\beta}}^\alpha \theta^{\bar{\beta}} \wedge \theta^n, \quad 2ig^{\bar{\nu}\alpha} h_{\bar{\nu}\bar{\beta}n} = h_{\bar{\beta}}^\alpha.$$

In view of (2.8), we can write

$$(2.18) \quad d \begin{pmatrix} \omega \\ \omega^\alpha \\ \omega^{\bar{\alpha}} \end{pmatrix} = \begin{pmatrix} \Delta & 0 & 0 \\ \Delta^\alpha & \Delta_\beta^\alpha & 0 \\ \Delta^{\bar{\alpha}} & 0 & \Delta_{\bar{\beta}}^{\bar{\alpha}} \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \omega^\beta \\ \omega^{\bar{\beta}} \end{pmatrix} + \begin{pmatrix} i\hat{g}_{\bar{\mu}\nu} \omega^{\bar{\mu}} \wedge \omega^\nu \\ \hat{h}_{\bar{\mu}}^\alpha \omega^{\bar{\mu}} \wedge \theta^n \\ \hat{h}_{\bar{\mu}}^\alpha \omega^\mu \wedge \theta^{\bar{n}} \end{pmatrix},$$

where $\Delta, \Delta^\alpha, \Delta^{\bar{\alpha}} := \overline{\Delta^\alpha}, \Delta_\beta^\alpha, \Delta_{\bar{\beta}}^{\bar{\alpha}} := \overline{\Delta_{\bar{\beta}}^{\bar{\alpha}}}$ are Maurer–Cartan forms for G modulo $\omega, \omega^\alpha, \omega^{\bar{\alpha}}, \theta^n, \theta^{\bar{n}}$, and where $\hat{g}_{\bar{\alpha}\beta}$ and $\hat{h}_{\bar{\beta}}^\alpha$ are functions on $Y \cong M \times G$ satisfying the following

$$(2.19) \quad u^{-1} \hat{g}_{\bar{\alpha}\beta}(p, ST) \overline{u_\mu^\alpha} u_\nu^\beta = \hat{g}_{\bar{\mu}\nu}(p, T), \quad \hat{h}_{\bar{\beta}}^\alpha(p, ST) \overline{u_\mu^\beta} = \hat{h}_{\bar{\mu}}^\nu(p, T) u_\nu^\alpha,$$

where $S, T \in G$ with S given by (2.6) and $p \in M$. The last action of G in (2.19) can also be described using the cubic tensor $\hat{h}_{\bar{\alpha}\bar{\beta}n} = \hat{g}_{\bar{\alpha}\mu} h_\alpha^\mu / 2i$ by

$$(2.20) \quad u^{-1} \hat{h}_{\bar{\alpha}\bar{\beta}n}(p, ST) \overline{u_\mu^\alpha} u_\nu^\beta = \hat{h}_{\bar{\mu}\bar{\nu}n}(p, T).$$

3. THE FIVE DIMENSIONAL CASE

We shall now proceed under the assumption that $n = 2$. Thus, M is a five dimensional CR manifold which is Levi uniform of rank 1; Greek indices $\alpha, \beta, \dots$ now run over the single integer 1. In what follows, we shall also assume that M is 2-nondegenerate at p_0 , which in this case simply amounts to requiring $\hat{h}_{\bar{1}\bar{1}2}(p_0) \neq 0$.

In view of the first identity in (2.19) and (2.20), we can make an initial choice of the θ, θ^1 so that (the 1×1 matrix) $g_{\bar{1}1}$ is constant with $g_{\bar{1}1} = 1$, and (the 1×1 matrix) $h_{\bar{1}\bar{1}2}$ is positive near p_0 . By a suitable initial choice of θ^2 , we may further assume that $h_{\bar{1}\bar{1}2}$ is constantly equal to 1 near p_0 . It follows from (2.19) and (2.20), as the reader can verify, that on the bundle Y the equations

$$(3.1) \quad \hat{g}_{\bar{1}1} = \hat{h}_{\bar{1}\bar{1}2} = 1$$

define a reduction (see e.g. [Ste, Chap. VII]) Y'_1 of Y to a principal bundle over a neighborhood of p_0 (which we shall identify with M) with group G'_1 consisting of those matrices S , as in (2.6) with $n = 2$, for which $u > 0$ and $u_1^1 = \pm u^{1/2}$. Hence,

the bundle Y'_1 has two connected components over M and for each component the covectors $\omega^1(p_0, u, u_1^1)$, given by the points on this component, define a linear form on $\bar{\mathcal{V}}_{p_0}$ which is invariantly defined up to multiplication by a *positive* number. We shall refer to a choice of this linear form (up to multiplication by a positive real number) as a choice of *orientation for the normalized CR structure* at p_0 . Thus, a choice of orientation determines a component Y_1 of Y'_1 which is a principal bundle over M with group $G_1 \subset G$, where G_1 consists of those matrices S , as in (2.6) with $n = 2$, for which $u > 0$ and $u^{-1/2}u_1^1 = 1$. In what follows we shall assume that such a choice of orientation has been made. We shall use the fiber coordinates $(u, u^1) \in \mathbb{R}_+ \times \mathbb{C}$ in the trivialization of Y_1 given by a choice of $\theta, \theta^1, \theta^2$ as described above.

It is easy to compute the Lie algebra $\mathfrak{g}_1$ of G_1 . We have $T \in \mathfrak{g}_1$ if and only if

$$(3.2) \quad T = \begin{pmatrix} 2v & 0 & 0 \\ v^1 & v & 0 \\ \bar{v}^1 & 0 & v \end{pmatrix}, \quad (v, v^a) \in \mathbb{R} \times \mathbb{C}.$$

Taking the pullbacks of all the forms $\omega, \omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}}, \Delta, \Delta^1, \Delta^{\bar{1}}$ to Y_1 , we see from formula (2.8) (with $S \in G_1$ now) that, pulled back to Y_1 , the matrix

$$(3.3) \quad \underline{\Delta} = \begin{pmatrix} \Delta & 0 & 0 \\ \Delta^1 & \Delta^{\bar{1}} & 0 \\ \Delta^{\bar{1}} & 0 & \Delta^{\bar{\bar{1}}} \end{pmatrix}$$

is valued in $\mathfrak{g}_1$ modulo $(\omega, \omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}})$, i.e.

$$(3.4) \quad \Delta^1 = \Delta^{\bar{1}} = \frac{1}{2}\Delta \quad \text{mod } (\omega, \omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}}).$$

Let us therefore rewrite (2.18) as follows

$$(3.5) \quad d \begin{pmatrix} \omega \\ \omega^1 \\ \omega^{\bar{1}} \end{pmatrix} = \begin{pmatrix} \Delta & 0 & 0 \\ \Delta^1 & \frac{1}{2}\Delta & 0 \\ \Delta^{\bar{1}} & 0 & \frac{1}{2}\Delta \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \omega^1 \\ \omega^{\bar{1}} \end{pmatrix} + \begin{pmatrix} i\omega^{\bar{1}} \wedge \omega^1 \\ \hat{t}_{11}^1 \omega^{\bar{1}} \wedge \omega^1 \\ \hat{t}_{11}^1 \omega^1 \wedge \omega^{\bar{1}} \end{pmatrix} + \begin{pmatrix} 0 \\ 2i\omega^{\bar{1}} \wedge \theta^2 + \hat{r}_1^1 \omega^1 \wedge \theta^2 + \hat{s}_1^1 \theta^{\bar{2}} \wedge \omega^1 \\ -2i\omega^1 \wedge \theta^{\bar{2}} + \hat{r}_1^1 \omega^{\bar{1}} \wedge \theta^{\bar{2}} + \hat{s}_1^1 \theta^2 \wedge \omega^{\bar{1}} \end{pmatrix},$$

where, by a slight abuse of notation, all the forms in (3.5) denote the pulled back forms on Y_1 . The 1-form Δ is uniquely determined by the form of the structure equation (3.5) up to transformations

$$(3.6) \quad \tilde{\Delta} = \Delta + a\omega,$$

where a is a smooth function on Y_1 ; i.e. replacing Δ by $\tilde{\Delta}$ given by (3.6) preserves (3.5) and this is the only transformation doing so, as is easily verified (cf. also [G, Lecture 3]). In fact, a direct calculation shows that

$$\Delta = u^{-1}du + iu^{-1/2}(\bar{u}^1\omega^1 - u^1\omega^{\bar{1}}) + \phi \quad \text{mod } \omega.$$

where ϕ is the pullback of a real 1-form on M . The 1-form θ^2 on Y_1 is determined by the condition $\hat{h}_{\bar{1}\bar{1}2} \equiv 1$ up to transformations

$$(3.7) \quad \tilde{\theta}^2 = \theta^2 + c_1 \omega^1 + c \omega,$$

where c, c_1 are smooth functions on Y_1 . The 1-form Δ^1 are determined modulo $\omega, \omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}}$. The precise form of the indeterminacy in Δ^1 is not important at this point.

By using the integrability of $\mathcal{V}$ and the fact that $\hat{h}_{\bar{1}\bar{1}2}$ is constant on Y_1 , we deduce that we can write

$$(3.8) \quad \begin{aligned} d\tilde{\theta}^2 &= d\theta^2 + dc_1 \wedge \omega^1 + c_1 d\omega^1 + dc \wedge \omega + cd\omega \\ &= \eta \wedge \omega + \eta_1 \wedge \omega^1 + \tilde{k}_{\bar{1}} \omega^{\bar{1}} \wedge \tilde{\theta}^2 + \hat{k} \tilde{\theta}^{\bar{2}} \wedge \tilde{\theta}^2, \end{aligned}$$

where η and η_1 are 1-forms (depending also on c_1 and c),

$$(3.9) \quad \tilde{k}_{\bar{1}} = \hat{k}_{\bar{1}} + 2ic_1 - \overline{c_1} \hat{k},$$

and $\hat{k}_{\bar{1}}, \hat{k}$ are smooth functions on Y_1 . Moreover, $\hat{k}$ is constant on the fibers $Y_1 \rightarrow M$, whereas $\hat{k}_1(p, ST) = u^{-1/2} \hat{k}_{\bar{1}}(p, T)$ for $S, T \in G_1$ and S of the form (2.6). The reader can easily verify that the function $\hat{k}$ is also uniquely determined, i.e. independent of the choice of $\Delta, \Delta^1, \theta^2$. It is also easy to check that the functions $\hat{r}_1^1, \hat{s}_1^1$ in (3.5) are uniquely determined, and that they are constant on the fibers $Y_1 \rightarrow M$. Hence, $\hat{k}(p), \hat{r}_1^1(p), \hat{s}_1^1(p)$ are invariants of the CR structure on M at $p \in M$. However, they are not independent as the following proposition shows.

Proposition 3.10. *We have the following identities*

$$(3.11) \quad \hat{s}_1^1 = \overline{\hat{r}_1^1} = \frac{1}{2} \hat{k}.$$

Proof. Differentiating the first row in the structure equation (3.5), using the fact that $d^2\omega = 0$, we obtain

$$(3.12) \quad 0 = d\Delta \wedge \omega - \Delta \wedge d\omega + i(d\omega^{\bar{1}} \wedge \omega^1 - \omega^{\bar{1}} \wedge d\omega^1).$$

Applying equation (3.5) again, we obtain

$$(3.13) \quad 0 = \left(d\Delta + i(\Delta^1 \wedge \omega^{\bar{1}} - \Delta^{\bar{1}} \wedge \omega^1) \right) \wedge \omega + i \left(\hat{s}_1^1 - \overline{\hat{r}_1^1} \right) \theta^{\bar{2}} \wedge \omega^{\bar{1}} \wedge \omega^1.$$

The first identity in (3.11) follows immediately from (3.13). To prove the second identity, we differentiate the formula for $d\omega^1$ given by (3.5). We obtain

$$(3.14) \quad \begin{aligned} 0 &= d\Delta^1 \wedge \omega - \Delta^1 \wedge d\omega + \frac{1}{2}(d\Delta \wedge \omega^1 - \Delta \wedge d\omega^1) + d\hat{t}_{\bar{1}1}^1 \wedge \omega^{\bar{1}} \wedge \omega^1 + \\ &\quad \hat{t}_{\bar{1}1}^1(d\omega^{\bar{1}} \wedge \omega^1 - \omega^{\bar{1}} \wedge d\omega^1) + 2i(d\omega^{\bar{1}} \wedge \theta^2 - \omega^{\bar{1}} \wedge d\theta^2) + d\hat{r}_1^1 \wedge \omega^1 \wedge \theta^2 + \\ &\quad \hat{r}_1^1(d\omega^1 \wedge \theta^2 - \omega^1 \wedge d\theta^2) + d\hat{s}_1^1 \wedge \theta^{\bar{2}} \wedge \omega^1 + \hat{s}_1^1(d\theta^{\bar{2}} \wedge \omega^1 - \theta^{\bar{2}} \wedge d\omega^1). \end{aligned}$$

If we substitute (3.5) and (3.9) in (3.14) and collect the $\omega^{\bar{1}} \wedge \theta^{\bar{2}} \wedge \theta^2$ -terms, we obtain

$$(3.15) \quad 2i\hat{k} = 2i\overline{\hat{r}}_1^1 + 2i\hat{s}_1^1,$$

which proves the second identity in (3.11). $\square$

For future reference, let us remark that the function $\hat{t}_{\bar{1}1}^1$ in (3.5) depend on the choice of θ^2 . For a fixed such choice, the functions $\hat{t}_{\bar{1}1}^1$ satisfy the following on Y_1

$$(3.16) \quad \hat{t}_{\bar{1}1}^1(p, ST) = \frac{\hat{t}_{\bar{1}1}^1(p, T)}{\sqrt{u}} + i \frac{3}{2} \frac{u^1}{u},$$

where $S, T \in G_1$ and S is given by (2.6). Let us also observe that replacing θ^2 by $\tilde{\theta}^2$, as given by (3.7), the function $\hat{t}_{\bar{1}1}^1$ change by

$$(3.17) \quad \hat{t}_{\bar{1}1}^1 \rightarrow \hat{t}_{\bar{1}1}^1 - (2ic_1 + \hat{s}_1^1 \overline{c_1}).$$

We shall distinguish two cases and begin by handling the most difficult one, which also contains the tube over the light cone as given by Example 0.2.1.

3.1. The case $\hat{k}(p_0) \neq 0$. First, since $\hat{k}(p_0) \neq 0$, there exists a small open neighborhood of p_0 in which $\hat{k} \neq 0$. We shall identify M with this neighborhood of p_0 in this subsection. Recall the formula in (3.9) for the coefficient $\tilde{k}_{\bar{1}}$ of $\omega^{\bar{1}} \wedge \tilde{\theta}^2$ in $d\tilde{\theta}^2$. Since $|\hat{k}(p_0)|$ could equal 2, it may not be possible to solve the equation $\tilde{k}_{\bar{1}} = 0$, i.e.

$$(3.1.1) \quad \hat{k}_{\bar{1}} + 2ic_1 - \overline{c_1} \hat{k} = 0,$$

for c_1 in a neighborhood of p_0 . Let us write

$$(3.1.2) \quad \hat{k} = 2ire^{it},$$

where $r > 0$ and t are real valued functions on Y_1 which are constant on the fibers of $Y_1 \rightarrow M$. We shall seek c_1 in the form

$$(3.1.3) \quad c_1 = e^{it/2}(\rho_1 + i\rho_2),$$

where $\rho_1, \rho_2 \in \mathbb{R}$. We can choose ρ_2 uniquely so that

$$\tilde{k}_{\bar{1}} = ir'e^{it/2},$$

for some real valued function r' on Y_1 which is constant on the fibers of $Y_1 \rightarrow M$. The function ρ_2 satisfies the same transformation rule as $\hat{k}_{\bar{1}}$, i.e.

$$(3.1.4) \quad \rho_2(p, ST) = u^{-1/2} \rho_2(p, T),$$

for $S, T \in G_1$ and S of the form (2.6). We can express the above by saying that $\text{Im } e^{-it/2} c_1$ is uniquely determined by the condition

$$(3.1.5) \quad \text{Re } e^{-it/2} \hat{k}_{\bar{1}} \equiv 0.$$

Now, with $\text{Im } e^{-it/2}c_1$ determined by (3.1.5), the equation (3.17) implies that the function $\hat{t}_{11}^1$ is determined up to

$$\hat{t}_{11}^1 \rightarrow \hat{t}_{11}^1 - \left(2i\rho_1 e^{it/2} + \hat{s}_1^1 \rho_1 e^{-it/2}\right)$$

or equivalently, in view of Proposition 3.10 and (3.1.2),

$$(3.1.6) \quad \begin{aligned} \hat{t}_{11}^1 &\rightarrow \hat{t}_{11}^1 - \left(2i\rho_1 e^{it/2} + \frac{1}{2} \hat{k} \rho_1 e^{-it/2}\right) \\ &= \hat{t}_{11}^1 - 2i\rho_1 e^{it/2} \left(1 + \frac{1}{2} r\right). \end{aligned}$$

Let us write $u^1 = e^{it/2}(x + iy)$. It follows from (3.16) that there is a submanifold $Y_2 \subset Y_1$, which is defined (uniquely in view of (3.1.6)) by the equation

$$(3.1.7) \quad \text{Re}(e^{-it/2} \hat{t}_{11}^1) = 0.$$

The manifold Y_2 can be viewed as a principal fiber bundle $Y_2 \rightarrow M$ with group $G_2 \subset G_1$, where G_2 is defined by the equation

$$(3.1.8) \quad \text{Im } e^{-it(p_0)/2} u^1 = 0,$$

if we let G_2 act on Y_2 as follows

$$(3.1.9) \quad g \begin{pmatrix} \omega \\ \omega^\alpha \\ \omega^{\bar{\alpha}} \end{pmatrix} := \begin{pmatrix} u & 0 & 0 \\ e^{it/2}x & \sqrt{u} & 0 \\ e^{-it/2}x & 0 & \sqrt{u} \end{pmatrix} \begin{pmatrix} \omega \\ \omega^\alpha \\ \omega^{\bar{\alpha}} \end{pmatrix},$$

for $g \in G_2 \subset G_1$ of the form

$$(3.1.10) \quad \begin{pmatrix} u & 0 & 0 \\ e^{it(p_0)/2}x & \sqrt{u} & 0 \\ e^{-it(p_0)/2}x & 0 & \sqrt{u} \end{pmatrix}.$$

Observe that the principal G_2 bundle Y_2 is a reduction of the principal G_1 bundle Y_1 only if the phase function t is constant. By definition, we have $\hat{t}_{11}^1 = ir''e^{it/2}$, for some real valued function r'' , on the bundle Y_2 . Hence, we can determine ρ_1 uniquely in (3.1.6) so that

$$(3.1.11) \quad \hat{t}_{11}^1 \equiv 0,$$

on Y_2 . It follows from (2.44) and (3.1.6) that we have

$$(3.1.12) \quad \rho_1(p, ST) = \frac{\rho_1(p, T)}{\sqrt{u}} + \frac{3}{2(2+r)} \frac{x}{u},$$

for $S, T \in G_2$ and S of the form (2.6) with $u^1 = e^{it/2}x$ and $x \in \mathbb{R}$.

Thus, θ^2 is determined on Y_2 up to

$$(3.1.13) \quad \tilde{\theta}^2 = \theta^2 + c\omega$$

by (3.1.5) and (3.1.11). Observe that on Y_2 , where $u^1 = e^{it/2}x$ for $x \in \mathbb{R}$, we have

$$(3.1.14) \quad \Delta^1 = e^{it/2}\xi \quad \text{mod } \omega, \omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}},$$

where ξ is a real 1-form on Y_2 which is not uniquely determined. We can write the structure equation for $d\omega^1$ on Y_2 as follows

$$(3.1.15) \quad d\omega^1 = e^{it/2}\xi \wedge \omega + \frac{1}{2}\Delta \wedge \omega^1 + 2i\omega^{\bar{1}} \wedge \theta^2 + \hat{r}_1^1 \omega^1 \wedge \theta^2 + \hat{s}_1^1 \theta^{\bar{2}} \wedge \omega^1 \\ + ie^{it/2}(b\omega^1 \wedge \omega + \bar{b}\omega^{\bar{1}} \wedge \omega + e\theta^2 \wedge \omega + \bar{e}\theta^{\bar{2}} \wedge \omega),$$

for some functions b and e on Y_2 . Using (3.6) and (3.1.13), we obtain

$$(3.1.16) \quad d\omega^1 = e^{it/2}\tilde{\xi} \wedge \omega + \frac{1}{2}\tilde{\Delta} \wedge \omega^1 + 2i\omega^{\bar{1}} \wedge \tilde{\theta}^2 + \hat{r}_1^1 \omega^1 \wedge \tilde{\theta}^2 + \hat{s}_1^1 \tilde{\theta}^{\bar{2}} \wedge \omega^1 \\ + ie^{it/2}(\tilde{b}\omega^1 \wedge \omega + \bar{\tilde{b}}\omega^{\bar{1}} \wedge \omega + e\tilde{\theta}^2 \wedge \omega + \bar{e}\tilde{\theta}^{\bar{2}} \wedge \omega),$$

where

$$(3.1.17) \quad \tilde{b} = b + \frac{1}{2} \left(\frac{1}{2}ae^{-it/2} + \bar{c}\hat{s}_1^1 e^{-it/2} - 2i\bar{c}e^{it/2} - c\hat{r}_1^1 e^{-it/2} \right)$$

and

$$(3.1.18) \quad \tilde{\xi} = \xi + \frac{1}{2} \left(\frac{1}{2}(ae^{-it/2}\omega^1 + \bar{a}e^{it/2}\omega^{\bar{1}}) + \bar{c}\hat{s}_1^1 e^{-it/2}\omega^1 + c\overline{\hat{s}_1^1} e^{it/2}\omega^{\bar{1}} \right) \\ - \frac{1}{2} \left(-2i\bar{c}e^{it/2}\omega^1 + 2ice^{-it/2}\omega^{\bar{1}} + c\hat{r}_1^1 e^{-it/2}\omega^1 + \bar{c}\overline{\hat{r}_1^1} e^{it/2}\omega^{\bar{1}} \right) + q\omega.$$

Here, q is an arbitrary real valued function on Y_2 , and a and c are as in (3.6) and (3.1.13) respectively. We deduce that ξ is determined by the structure equation (3.1.16) up to transformations given by (3.1.18). Let us rewrite (3.1.17) and (3.1.18) using Proposition 3.10 and (3.1.2) as follows

$$(3.1.19) \quad \tilde{b} = b + \frac{1}{2}e^{-it/2} \left(\frac{1}{2}a + i(r-2)\bar{\zeta} + ir\zeta \right),$$

and

$$(3.1.20) \quad \tilde{\xi} = \xi + \frac{1}{2}e^{-it/2} \left(\frac{1}{2}a + i(r+2)\bar{\zeta} + ir\zeta \right) \omega^1 \\ + \frac{1}{2}e^{it/2} \left(\frac{1}{2}a - i(r+2)\zeta - ir\bar{\zeta} \right) \omega^{\bar{1}} + q\omega,$$

where we have used the notation

$$\zeta = e^{-it}c.$$

By (3.13), we have on Y_1 ,

$$(3.1.21) \quad d\Delta = i(\Delta^{\bar{1}} \wedge \omega^1 - \Delta^1 \wedge \omega^{\bar{1}}) + \Phi \wedge \omega,$$

for some real 1-form Φ . Hence, on Y_2 we have, by (3.1.14),

$$(3.1.22) \quad d\Delta = ie^{-it/2}\xi \wedge \omega^1 - ie^{it/2}\xi \wedge \omega^{\bar{1}} + \Psi \wedge \omega + \Psi_1 \wedge \omega^1 + \Psi_{\bar{1}} \wedge \omega^{\bar{1}},$$

for some 1-forms $\Psi, \Psi_1, \Psi_{\bar{1}} = \overline{\Psi_1}$ on Y_2 such that $\Psi_1 = 0$ modulo $\omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}}$. Recall that Δ is determined up to transformations (3.6), θ^2 up to transformations (3.1.13), and ξ up to transformations (3.1.20). Substituting in (3.1.22), we obtain

$$(3.1.23) \quad \begin{aligned} d\tilde{\Delta} &= d\Delta + da \wedge \omega + ad\omega \\ &= ie^{-it/2}\tilde{\xi} \wedge \omega^1 - ie^{it/2}\tilde{\xi} \wedge \omega^{\bar{1}} + i\tilde{f}\omega^{\bar{1}} \wedge \omega^1 + \dots, \end{aligned}$$

where $\dots$ signify the remaining terms in the expansion of $d\tilde{\Delta}$ and

$$(3.1.24) \quad \tilde{f} = f - \frac{1}{2}a + i\zeta - i\bar{\zeta},$$

for some real valued function f on Y_2 . Hence, a is uniquely determined as a function of ζ by the condition

$$(3.1.25) \quad \tilde{f} = 0,$$

and we have

$$(3.1.26) \quad a = 2i(\zeta - \bar{\zeta}).$$

In view of (3.1.3), (3.1.4), and (3.1.12), we have

$$(3.1.27) \quad dc_1 = e^{it/2} \frac{3}{2(2+r)} \xi \quad \text{mod } \Delta, \omega, \omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}}.$$

It follows, by using (3.8), (3.1.20) and (3.1.24), that

$$(3.1.28) \quad d\tilde{\theta}^2 = e^{it/2} \frac{3}{2(2+r)} \tilde{\xi} \wedge \omega^1 + \tilde{m}\omega^{\bar{1}} \wedge \omega^1 + \dots,$$

where $\dots$ signify the remaining terms in the expansion of $d\tilde{\theta}^2$ and $\tilde{m}$ is given by

$$(3.1.29) \quad \begin{aligned} \tilde{m} &= m + \frac{3ie^{it}}{4} \frac{(r+1)}{(r+2)} (\zeta + \bar{\zeta}) + ic \\ &= m + \frac{ie^{it}}{4(r+2)} ((7r+11)\zeta + 3(r+1)\bar{\zeta}) \end{aligned}$$

for some function m on Y_2 ; in the last line of (3.1.29), we have used $c = e^{it}\zeta$. Since $r > 0$, we can determine ζ uniquely by the condition

$$(3.1.30) \quad \tilde{m} = 0.$$

Let us summarize our efforts so far. We have determined Δ and θ^2 uniquely on Y_2 . The 1-form ξ is determined up to

$$(3.1.31) \quad \tilde{\xi} = \xi + q\omega,$$

for some real valued function q on Y_2 . We shall conclude the construction in this section by defining a unique choice of ξ . In view of (3.1.22), we have

$$(3.1.32) \quad d\Delta = ie^{-it/2}\tilde{\xi} \wedge \omega^1 - ie^{it/2}\tilde{\xi} \wedge \omega^{\bar{1}} + \tilde{l}\omega^1 \wedge \omega + \bar{\tilde{l}}\omega^{\bar{1}} \wedge \omega + \dots,$$

where

$$(3.1.33) \quad \tilde{l} = l + ie^{-it/2}q$$

for some function l on Y_2 . Hence, we may determine q uniquely by the condition

$$(3.1.34) \quad \operatorname{Im} e^{it/2}\tilde{l} = 0.$$

The uniquely determined, linearly independent 1-forms

$$(3.1.35) \quad \omega, \operatorname{Re} \omega^1, \operatorname{Im} \omega^1, \operatorname{Re} \theta^2, \operatorname{Im} \theta^2, \Delta, \xi$$

on Y_2 form a global coframe for T^*Y_2 and, hence, define an absolute parallelism on Y_2 . The following result is a consequence of the construction above (see [G] or [CM]). We use the notation introduced previously, and also

$$(3.1.36) \quad \underline{\omega} := \begin{pmatrix} \omega \\ \operatorname{Re} \omega^1 \\ \operatorname{Im} \omega^1 \\ \operatorname{Re} \theta^2 \\ \operatorname{Im} \theta^2 \\ \Delta \\ \xi \end{pmatrix}.$$

Theorem 3.1.37. *Let M be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0 \in M$. Suppose that $\hat{k}(p_0) \neq 0$ and that an orientation is chosen for the normalized CR structure at p_0 (as explained in §3). Then, there exist an open neighborhood M_0 of p_0 in M , a principal fiber bundle $Y_2 \rightarrow M_0$ with a two dimensional structure group $G_2 \subset GL(\mathbb{C}^3)$ and a 1-form $\underline{\omega}$ on Y_2 which defines an isomorphism between $T_y Y_2$ and $\mathbb{R}^7$ for every $y \in Y_2$ and such that the following holds. Let M' be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p'_0 \in M$. Suppose that $\hat{k}'(p'_0) \neq 0$ (where corresponding objects for M' are denoted with ') and that an orientation is chosen for the normalized CR structure of M' at p'_0 . Then, if there exists a local CR diffeomorphism $f: (M, p_0) \rightarrow (M', p'_0)$ preserving the oriented CR structures of M and M' at p_0 and p'_0 , respectively, there exists a diffeomorphism $F: Y_2 \rightarrow Y'_2$ with $\pi' \circ F(\pi^{-1}(p_0)) = \{p'_0\}$ such that $F^*\underline{\omega}' = \underline{\omega}$ and the following diagram commutes*

$$(3.1.38) \quad \begin{array}{ccc} Y_2 & \xrightarrow{F} & Y'_2 \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{f} & M'. \end{array}$$

Conversely, if there exists a diffeomorphism $F: Y_2 \rightarrow Y'_2$ with $\pi' \circ F(\pi^{-1}(p_0)) = p'_0$ such that $F^\underline{\omega}' = \underline{\omega}$, then there exists a CR diffeomorphism $f: (M, p_0) \rightarrow (M', p'_0)$ preserving the oriented CR structures of M and M' at p_0 and p'_0 , respectively, such that (3.1.38) commutes.*

The 1-form $\underline{\omega}$ is given by (3.1.36) and is uniquely determined by (3.1.5), (3.1.11), (3.1.25), (3.1.30), and (3.1.34). The group G_2 is defined by (3.1.10).

3.2. The case $|\hat{k}(p_0)| \neq 2$. Since $|\hat{k}(p_0)| \neq 2$, there is a small neighborhood of p_0 in M in which $|\hat{k}| \neq 2$. In this neighborhood, which we identify with M in this subsection, we can solve uniquely for c_1 in the equation

$$(3.2.1) \quad \tilde{k}_{\bar{1}} = 0,$$

where $\tilde{k}_{\bar{1}}$ is given by equation (3.1.1). Thus, c_1 is uniquely determined by the condition (3.2.1), and hence θ^2 is determined up to transformations of the form (3.1.13). Moreover, it follows from (3.16) that there exists a uniquely determined submanifold $Y_3 \subset Y_1$ defined by

$$(3.2.2) \quad \hat{t}_{11}^1 = 0.$$

The submanifold Y_3 is a subbundle (a reduction) of the principal G_1 -bundle Y_1 with group G_3 , where G_3 is the subgroup of G_1 defined by

$$(3.2.3) \quad u^1 = 0.$$

Thus, on Y_3 we have $\Delta^1 = 0$ modulo $\omega, \omega^1, \omega^{\bar{1}}, \theta^2, \theta^{\bar{2}}$. It follows that the structure equation for $d\omega^1$ on Y_3 can be written

$$(3.2.4) \quad d\omega^1 = \frac{1}{2}\tilde{\Delta} \wedge \omega^1 + 2i\omega^{\bar{1}} \wedge \tilde{\theta}^2 + \hat{r}_1^1 \omega^1 \wedge \tilde{\theta}^2 + \hat{s}_1^1 \tilde{\theta}^{\bar{2}} \wedge \omega^1 \\ + {}^1\tilde{b}\omega^1 \wedge \omega + {}^2b\tilde{\omega}^{\bar{1}} \wedge \omega + {}^1e\theta^2 \wedge \omega + {}^2e\theta^{\bar{2}} \wedge \omega,$$

where

$$(3.2.5) \quad {}^1\tilde{b} := {}^1b - \hat{r}_1^1 c + \hat{s}_1^1 \bar{c} + \frac{1}{2}a, \quad {}^2\tilde{b} := {}^2b - 2ic$$

for some functions 1b , 2b , 1e , and 2e on Y_3 . We can determine c , and hence θ^2 , uniquely on Y_3 by the condition

$$(3.2.6) \quad {}^2\tilde{b} = 0.$$

We then determine a , and hence Δ , uniquely on Y_3 by the condition

$$(3.2.7) \quad \operatorname{Re} {}^1\tilde{b} = 0.$$

The uniquely determined 1-forms

$$\omega, \operatorname{Re} \omega^1, \operatorname{Im} \omega^1, \operatorname{Re} \theta^2, \operatorname{Im} \theta^2, \Delta$$

on Y_3 form a global coframe for T^*Y_3 and, hence, define an absolute parallelism on Y_3 . As in 3.1, we have the following result. We use the notation introduced above, and also

$$(3.2.8) \quad \underline{\omega} := \begin{pmatrix} \omega \\ \operatorname{Re} \omega^1 \\ \operatorname{Im} \omega^1 \\ \operatorname{Re} \theta^2 \\ \operatorname{Im} \theta^2 \\ \Delta \end{pmatrix}$$

Theorem 3.2.9. *Let M be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0 \in M$. Suppose that $|\hat{k}(p_0)| \neq 2$ and that an orientation is chosen for the normalized CR structure at p_0 (as explained in §3). Then, there exist an open neighborhood M_0 of p_0 in M , a principal fiber bundle $Y_3 \rightarrow M_0$ with a one dimensional structure group $G_3 \subset GL(\mathbb{C}^3)$ and a 1-form $\underline{\omega}$ on Y_3 which defines an isomorphism between $T_y Y_3$ and $\mathbb{R}^6$ for every $y \in Y_3$ and such that the following holds. Let M' be a 5-dimensional CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p'_0 \in M$. Suppose that the invariant $|\hat{k}'(p'_0)| \neq 2$ (where corresponding objects for M' are denoted with $'$) and that an orientation is chosen for the normalized CR structure of M' at p'_0 . Then, if there exists a local CR diffeomorphism $f: (M, p_0) \rightarrow (M', p'_0)$ preserving the oriented CR structures of M and M' at p_0 and p'_0 , respectively, there exists a diffeomorphism $F: Y_3 \rightarrow Y'_3$ with $\pi' \circ F(\pi^{-1}(p_0)) = \{p'_0\}$ such that $F^* \underline{\omega}' = \underline{\omega}$ and the following diagram commutes*

$$(3.2.10) \quad \begin{array}{ccc} Y_3 & \xrightarrow{F} & Y'_3 \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{f} & M'. \end{array}$$

Conversely, if there exists a diffeomorphism $F: Y_3 \rightarrow Y'_3$ with $\pi' \circ F(\pi^{-1}(p_0)) = p'_0$ such that $F^* \underline{\omega}' = \underline{\omega}$, then there exists a CR diffeomorphism $f: (M, p_0) \rightarrow (M', p'_0)$ preserving the oriented CR structures of M and M' , such that (3.2.10) commutes.

The 1-form $\underline{\omega}$ is given by (3.2.8) and is uniquely determined by (3.2.1), (3.2.6), and (3.2.7). The group G_3 is defined by (3.2.3), and on Y_3 (3.2.2) holds.

4. A CURVATURE CHARACTERIZATION OF THE TUBE OVER THE LIGHT CONE IN $\mathbb{C}^3$

In this section, we shall continue to consider only the case $n = 2$. We shall change slightly the convention from section 2 that capital Roman indices A, B , etc. run over the set $\{1, 2\}$ (i.e. $\{1, \dots, n\}$ with $n = 2$) and instead let them run over the set $\{1, 2, 3\}$. We shall also use the convention that small Roman indices a, b , etc. run over the set $\{0, 1, 2, 3\}$.

Denote by Γ the light cone in $\mathbb{R}^3$, i.e. the zero locus of the quadratic form $\{x, x\}$ where $\{\cdot, \cdot\}$ denotes the bilinear form which in the standard coordinates of $\mathbb{R}^3$ is given by

$$(4.1) \quad \{x, y\} := x^1 y^1 + x^2 y^2 - x^3 y^3.$$

We shall denote by $\Gamma_{\mathbb{C}}$ the tube in $\mathbb{C}^3$ over Γ (as in Example 0.2.1). Hence, $\Gamma_{\mathbb{C}}$ is the zero locus of $\{Z, Z\}_{\mathbb{C}}$, where $\{z, w\}_{\mathbb{C}}$ for complex vectors $z, w \in \mathbb{C}^3$ is defined by

$$(4.2) \quad \{z, w\}_{\mathbb{C}} := \{\operatorname{Re} z, \operatorname{Re} w\}.$$

We shall call a *frame* for $\Gamma_{\mathbb{C}}$ a 4-tuple (Z_0, Z_1, Z_2, Z_3) of 4-vectors, where

$$(4.3) \quad Z_0 = t(1, z_0), \quad Z_A = (0, x_A), \quad t \in \mathbb{R}, \quad z_0 \in \mathbb{C}^3, \quad x_A \in \mathbb{R}^3,$$

which satisfy the following conditions. The real vectors $x_1, x_2, x_3 \in \mathbb{R}^3$ satisfy

$$(4.4) \quad \begin{aligned} \{x_1, x_1\} &= \{x_3, x_3\} = \{x_1, x_2\} = \{x_2, x_3\} = 0, \\ \{x_1, x_3\} &= -1, \quad \{x_2, x_2\} = 1 \end{aligned}$$

and also,

$$(4.5) \quad \operatorname{Re} tz_0 = x_1.$$

Observe that the conditions (4.4) are equivalent to the fact that the (symmetric) matrix representation of the bilinear form $\{\cdot, \cdot\}$ relative to the basis x_1, x_2, x_3 is by the matrix Λ , where

$$(4.6) \quad \Lambda := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

We also write

$$(4.7) \quad \Lambda = (\lambda_{AB}).$$

The set of all frames for $\Gamma_{\mathbb{C}}$ can be viewed as a real subgroup of the complex Lie group $GL(\mathbb{C}^4)$ as follows. If (Z_0, Z_A) is a given frame, then any other frame (Z'_0, Z'_A) is obtained as

$$(4.8) \quad (Z'_0, Z'_A) = (Z_0, Z_B) \begin{pmatrix} v & 0 \\ (k_1^B - v\delta_1^B) + iv^B & k_A^B \end{pmatrix},$$

where $v, v^B \in \mathbb{R}$ and the 3×3 -matrix (k_A^B) satisfies

$$(4.9) \quad k_A^C k_B^D \lambda_{CD} = \lambda_{AB}$$

and also

$$(4.10) \quad \det(k_B^A) = 1.$$

We denote by H' the subgroup of $GL(\mathbb{C}^4)$ consisting of all matrices of the form

$$(4.11) \quad \mathfrak{M} = \begin{pmatrix} v & 0 \\ (k_1^B - v\delta_1^B) + iv^B & k_A^B \end{pmatrix},$$

which satisfy (4.9) and (4.10). We also denote by K the subgroup of $GL(\mathbb{R}^3)$ which consists of (k_B^A) satisfying (4.9) and (4.10). The group K is isomorphic to the Lorentz group $SO(2, 1)$. Indeed, if O denote the orthogonal transformation for which $O^T \Lambda O$ equals the diagonal matrix with 1, 1, -1 on the diagonal, then $K = O(SO(2, 1))O^T$.

Note that (4.2), (4.4), and (4.5) imply that Z_0 , considered as the affine point $z_0 \in \mathbb{C}^3$ where Z_0 and z_0 are as in (4.3), can be viewed as a point on $\Gamma_{\mathbb{C}}$. We denote by H'_0 the subgroup of H' consisting of those matrices which preserve Z_0 as an affine point on $\Gamma_{\mathbb{C}}$, i.e. the group of matrices of the form

$$(4.12) \quad \mathfrak{N} = \begin{pmatrix} v & 0 \\ 0 & k_A^B \end{pmatrix},$$

where $(k_B^A) \in K$ with $k_1^A x_A = vx_1$. A straightforward calculation shows that (k_B^A) must be of the form

$$(4.13) \quad (k_B^A) = \begin{pmatrix} v & a & \frac{1}{2}a^2v^{-1} \\ 0 & 1 & av^{-1} \\ 0 & 0 & v^{-1} \end{pmatrix},$$

for some $a \in \mathbb{R}$. Thus, the group of frames for $\Gamma_{\mathbb{C}}$ (which, given a fixed frame, can be identified with the group H' via (4.8)) is a principal fiber bundle $P' \rightarrow \Gamma_{\mathbb{C}}$ with group H'_0 . Let us now choose an orientation, as explained in §3, for the normalized CR structure of $\Gamma_{\mathbb{C}}$ and denote by P the group of frames consistent with this orientation. Then, as is easily verified, P is isomorphic to the group H , where H is the subgroup of H' consisting of matrices of the form (4.11) with $v > 0$. If we also denote by H_0 the subgroup of H'_0 consisting of those matrices of the form (4.12) for which $v > 0$, then $P \rightarrow M$ is a principal fiber bundle with group H_0 . The reader should note that $\Gamma_{\mathbb{C}}$ has two connected components. This is reflected on the bundle P by the fact that the Lorentz group $SO(2, 1)$ has two components.

A choice for the 4×4 -matrix $\Pi = (\pi_b^a)$ of Maurer-Cartan forms for the group H is given by

$$(4.14) \quad dZ_a = \pi_a^b Z_b.$$

The Maurer-Cartan equations of structure then become

$$(4.15) \quad d\pi_a^b = \pi_a^c \wedge \pi_c^b,$$

which follows directly from differentiating (4.14). Due to the form (4.3) of the frames (Z_0, Z_A) , we have $\pi_0^0 = dv/v$, $\pi_A^0 = 0$, and the π_a^b are real. By differentiating the defining equations (4.5) and using again these equations, we deduce that the 3×3 -matrix (π_A^B) (which is a Maurer-Cartan matrix for the group K) is given by

$$(4.16) \quad (\pi_A^B) = \begin{pmatrix} \pi_1^1 & \pi_2^1 & 0 \\ \pi_1^2 & 0 & \pi_2^1 \\ 0 & \pi_1^2 & -\pi_1^1 \end{pmatrix},$$

for some real 1-forms $\pi_1^1, \pi_2^1, \pi_1^2$. By differentiating $Z_0 = v(1, z_0)$, we obtain

$$(4.17) \quad dZ_0 = \frac{dv}{v}(v, vz_0) + v(0, dz_0) = \frac{dv}{v}Z_0 + (0, vdz_0).$$

Thus, by equation (4.5), we have

$$(4.18) \quad \frac{dv}{v}x_1 + \frac{1}{2}(vdz_0 + vdz_0) = dx_1.$$

and hence, using also (4.14),

$$(4.19) \quad \frac{1}{2}(\pi_0^A + \bar{\pi}_0^A)x_A = \left(\pi_1^A - \frac{dv}{v}\delta_1^A \right) x_A.$$

Using (4.16), we obtain the equations

$$(4.20) \quad \begin{cases} \pi_1^1 - \pi_0^0 = \frac{1}{2}(\pi_0^1 + \bar{\pi}_0^1) \\ \pi_1^2 = \frac{1}{2}(\pi_0^2 + \bar{\pi}_0^2) \\ 0 = \pi_0^3 + \bar{\pi}_0^3. \end{cases}$$

The last formula in (4.20) implies that π_0^3 is a purely imaginary form. The matrix of 1-forms $\Pi = (\pi_b^a)$ is valued in the Lie algebra $\mathfrak{h}$ of H , and if we change frame using (4.8) then the corresponding matrix of 1-forms Π' for the new frame (Z'_0, Z'_A) is related to Π by

$$(4.21) \quad \Pi' = \text{ad } (\mathfrak{M}^{-1})\Pi := \mathfrak{M}^{-1}\Pi\mathfrak{M},$$

where $\mathfrak{M}$ is the matrix given by (4.11). It follows that Π is a Cartan connection on P with group H (see e.g. [K, Chapter IV] and also [CM]), which is flat (i.e. with vanishing curvature form) by (4.15).

Let us relate the above to the results in previous sections. It is straightforward to verify that we can set

$$(4.22) \quad \begin{cases} \omega = -\frac{i}{2}\pi_0^3 \\ \omega^1 = \frac{1}{2i}\pi_0^2 \\ \theta^2 = \frac{1}{4i}\pi_0^1 \\ \Delta = \pi_0^0 + \pi_1^1 = 2\pi_0^0 + \frac{1}{2}(\pi_0^1 + \bar{\pi}_0^1) \\ \xi(= \Delta^1) = -\pi_2^1, \end{cases}$$

for the forms given by Theorem 3.1.37. Indeed, using (4.15) and (4.20), we obtain the equations

$$(4.23) \quad \begin{cases} d\omega = \Delta \wedge \omega + i\omega^{\bar{1}} \wedge \omega^1 \\ d\omega^1 = \xi \wedge \omega + \frac{1}{2}\Delta \wedge \omega^1 - i\omega^1 \wedge \theta^2 + i\theta^{\bar{2}} \wedge \omega^1 + \omega^{\bar{1}} \wedge \theta^2 \\ d\theta^2 = \frac{1}{2}\xi \wedge \omega^1 + 2i\theta^{\bar{2}} \wedge \theta^2 \\ d\Delta = i\xi \wedge \omega^1 - i\xi \wedge \omega^{\bar{1}} \\ d\xi = -\frac{1}{2}(\Delta + 2i\theta^2 - 2i\theta^{\bar{2}}) \wedge \xi, \end{cases}$$

which satisfy the conditions of Theorem 3.1.37. Note that the invariant $\hat{k}$, defined in section 3, satisfies $\hat{k} \equiv 2i$. Thus, in what follows, the phase function t and the modulus r as defined by (3.1.2), are identically 0 and 1, respectively.

Recall that the group of frames is a principal fiber bundle $P \rightarrow \Gamma_{\mathbb{C}}$ with group H_0 . For any $\mathfrak{N} \in H_0$, where $\mathfrak{N} \in H_0$ is given by (4.12) and (k_B^A) by (4.13), a change of frame $(Z'_0, Z'_A) = (Z_0, Z_B)\mathfrak{N}$ results in the change of connection form

$$(4.24) \quad \Pi' = \text{ad } (\mathfrak{N}^{-1})\Pi = \begin{pmatrix} \pi_0^0 & 0 \\ \nu l_A^B \pi_0^A & l_A^B k_D^C \pi_C^A \end{pmatrix},$$

where (l_D^C) denotes the inverse of (k_B^A) . Using (4.22) and calculating the inverse of (k_A^B) given by (4.13), we deduce that (4.24) yields the corresponding transformation

$$(4.25) \quad \begin{cases} \omega' = v^2 \omega \\ (\omega^1)' = -av\omega + v\omega^1 \\ (\theta^2)' = \frac{1}{4}a^2\omega - \frac{1}{2}a\omega^1 + \theta^2 \\ \Delta' = iav^{-1}(\omega^{\bar{1}} - \omega^1) + \Delta \\ \xi' = \frac{1}{2}ia^2v^{-1}(\omega^1 - \omega^{\bar{1}}) - iav^{-1}(\theta^2 - \theta^{\bar{2}}) - \frac{1}{2}av^{-1}\Delta + v^{-1}\xi. \end{cases}$$

In particular, we obtain

$$(4.26) \quad \begin{pmatrix} \omega' \\ (\omega^1)' \\ (\omega^{\bar{1}})' \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ x & \sqrt{u} & 0 \\ x & 0 & \sqrt{u} \end{pmatrix} \begin{pmatrix} \omega \\ \omega^1 \\ \omega^{\bar{1}} \end{pmatrix},$$

where

$$(4.27) \quad u = v^2, \quad x = -av.$$

Hence, we have defined an isomorphism $\phi: H_0 \rightarrow G_2$, where G_2 is as defined in section 3.1 with $t \equiv 1$, defined by

$$(4.28) \quad \phi \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & v & a & \frac{1}{2}a^2v^{-1} \\ 0 & 0 & 1 & av^{-1} \\ 0 & 0 & 0 & v^{-1} \end{pmatrix} = \begin{pmatrix} v^2 & 0 & 0 \\ -av & v & 0 \\ -av & 0 & v \end{pmatrix}.$$

Let M be any 5-dimensional CR manifold of hypersurface type which satisfies the conditions in Theorem 3.1.37. Furthermore, we assume that the invariant $\hat{k} \equiv 2i$ in a neighborhood of p_0 . Let $\omega, \omega^1, \theta^2, \Delta, \xi$ be the forms given by Theorem 3.1.37 such that $\underline{\omega}$ is given by (3.1.36). Define the $\mathfrak{h}$ -valued 1-form $\Pi = (\pi_a^b)$ by

$$(4.29) \quad \Pi := \begin{pmatrix} \pi_0^0 & 0 & 0 & 0 \\ \pi_0^1 & \pi_1^1 & \pi_2^1 & 0 \\ \pi_0^2 & \pi_1^2 & 0 & \pi_2^2 \\ \pi_0^3 & 0 & \pi_1^2 & -\pi_1^1 \end{pmatrix},$$

where

$$\pi_0^0, \pi_0^1, \pi_0^2, \pi_0^3, \pi_1^1, \pi_1^2, \pi_2^1,$$

are obtained by solving (4.22), using also the first two equation of (4.20). Clearly, Π defines an isomorphism between $T_y Y_2$, where $Y_2 \rightarrow M$ is the principal bundle given by Theorem 3.1.37, and $\mathfrak{h}$ for every $y \in Y_2$. However, it is not difficult to verify that Π is in general not a Cartan connection, i.e. it does not transform according to (4.21). Nevertheless, by defining the curvature

$$(4.30) \quad \Omega = d\Pi - \Pi \wedge \Pi,$$

a direct consequence of Cartan's solution of the equivalence problem for $\{1\}$ - structures (see e.g. [G]) is the following characterization of the tube over the light cone.

Theorem 4.31. *Let M be a 5-dimensional real-analytic CR manifold of hypersurface type which is 2-nondegenerate and Levi uniform of rank 1 at $p_0 \in M$. Assume that $\hat{k} \equiv 2i$. Choose an orientation for the normalized CR structure at p_0 (as explained in §3), and denote by $Y_2 \rightarrow M_0$ the principal bundle over a neighborhood M_0 of p_0 , with 1-form $\underline{\omega}$, given by Theorem 3.1.37. Then, the $\mathfrak{h}$ -valued 1-form Π defined by (4.29), where*

$$(4.32) \quad \begin{cases} \pi_0^0 := \frac{1}{2}(\Delta - 2i\theta^2 + 2i\theta^{\bar{2}}) \\ \pi_1^1 := \frac{1}{2}(\Delta + 2i\theta^2 - 2i\theta^{\bar{2}}) \\ \pi_0^1 := 4i\theta^2 \\ \pi_0^2 := 2i\omega^1 \\ \pi_0^3 := 2i\omega \\ \pi_1^2 := i(\omega^1 - \omega^{\bar{1}}) \\ \pi_2^1 := -\xi \end{cases}$$

and $\underline{\omega}$ is given by (3.1.36), defines an isomorphism $T_y Y_2 \cong \mathfrak{h}$ for every $y \in Y_2$, with the following property. There exists a local real-analytic CR diffeomorphism $f: M \rightarrow \Gamma_{\mathbb{C}}$ near p_0 if and only if the curvature Ω given by (4.30) vanishes identically.

We conclude the discussion of the tube over the light cone by computing the dimension of the stability group $\text{Aut}(\Gamma_{\mathbb{C}}, p_0)$ of $\Gamma_{\mathbb{C}}$ at a point $p_0 \in \Gamma_{\mathbb{C}}$. Observe that, given a frame (Z_0, Z_A) in P , the manifold $\Gamma_{\mathbb{C}}$ can be viewed as the quotient group H/H_0 via the identification $P \cong H$ provided by (4.8). Let us denote the affine point on $\Gamma_{\mathbb{C}}$ corresponding to Z_0 by p_0 . Then, under the identification $\Gamma_{\mathbb{C}} \cong H/H_0$, p_0 corresponds to the coset eH_0 , where $e \in H$ denotes the identity matrix. The group H_0 acts on the left on H/H_0 and each homomorphism $aH_0 \mapsto baH_0$, for $b \in H_0$, preserves the point $p_0 \cong eH_0$. It is straightforward to verify that the action is effective; i.e. if, for $b \in H_0$, the homomorphism $aH_0 \mapsto baH_0$ is the identity, then $b = e$. Let us denote by $f_b: (\Gamma_{\mathbb{C}}, p_0) \rightarrow (\Gamma_{\mathbb{C}}, p_0)$ the mapping corresponding to the homomorphism $aH_0 \mapsto baH_0$. Each f_b is a CR diffeomorphism. (Indeed, it is not difficult to compute f_b in coordinates and see that f_b is induced by an invertible linear transformation of $\mathbb{C}^3$.) Thus, $b \mapsto f_b$ embeds H_0 as a subgroup of $\text{Aut}(\Gamma_{\mathbb{C}}, p_0)$. Since $\dim H_0 = 2$, we conclude that $\dim \text{Aut}(\Gamma_{\mathbb{C}}, p_0) \geq 2$. On the other hand, by Theorem 3.1.37 and [K, Theorem 3.2], it follows (as in the introduction) that the subgroup of $\text{Aut}(\Gamma_{\mathbb{C}}, p_0)$ consisting of those CR diffeomorphisms that preserve the orientation of the CR structure chosen above embeds as a closed submanifold of $P_{p_0} \cong H_0$. Hence, we have $\dim \text{Aut}(\Gamma_{\mathbb{C}}, p_0) = 2$.

5. PROOFS OF THEOREMS 4, 5, AND AN EXAMPLE

5.1. Proof of Theorem 4. First of all, since M, M' are assumed 2-nondegenerate and Levi uniform at $p_0 \in M, p'_0 \in M'$, it follows (as the reader can verify) that they must in fact be Levi uniform of rank one, and not of rank two (in which case they would be Levi nondegenerate) or zero (in which case they would be Levi flat). Thus, we may apply the main results of the present paper. We shall keep the notation

used in §3, and the convention that M and M' are assumed to be sufficiently small neighborhoods of p_0 and p'_0 , respectively. We have two cases to consider, namely $\hat{k}(p_0) \neq 0$ and $|\hat{k}(p_0)| \neq 2$. We give here only the proof in the former case and leave the other case (which is completely analogous) to the reader.

For brevity, we denote by G the group G_2 and by $P \rightarrow M$, $P' \rightarrow M'$, the principal G -bundles $Y_2 \rightarrow M$, $Y'_2 \rightarrow M'$ given by Theorem 3.1.37 (where some choice of orientations for the normalized CR structures of M and M' at p_0 and p'_0 , respectively, have been made). Let $\theta, \theta^1, \theta^2$ and $\theta', \theta'^1, \theta'^2$ be 1-forms on M and M' near p_0 and p'_0 , respectively, such that the principal bundles $P \rightarrow M$, $P' \rightarrow M'$ are locally trivialized: $P \cong M \times G$, $P' \cong M' \times G$.

Now, suppose that $f: (M, p_0) \rightarrow (M', p'_0)$ is a CR diffeomorphism. (There is no loss of generality in assuming that f preserves the orientations of the CR structures chosen above.) For the convenience of the reader, we give here an explicit description of the diffeomorphism $F: P \rightarrow P'$ provided by Theorem 3.1.37. Since f is a CR diffeomorphism, the 1-forms

$$(5.1.1) \quad \tilde{\theta} := f^*(\theta'), \quad \tilde{\theta}^\alpha := f^*((\theta^\alpha)'), \quad \alpha = 1, 2,$$

yield a basis for the sections of $T'M$. Moreover, $\tilde{\theta}, \tilde{\theta}^1$ yield a basis for the sections of $T''M$ and $\tilde{\theta}$ is a characteristic form near p_0 . In addition, there exists a smooth function γ near p_0 , valued in G , such that

$$(5.1.2) \quad \begin{pmatrix} \tilde{\theta} \\ \tilde{\theta}^1 \\ \tilde{\theta}^{\bar{1}} \end{pmatrix} = \gamma \begin{pmatrix} \theta \\ \theta^1 \\ \theta^{\bar{1}} \end{pmatrix}.$$

The mapping $F: P \rightarrow P'$ is defined by

$$(5.1.3) \quad F(p, S) := (f(p), S\gamma(p)^{-1}), \quad (p, S) \in M \times G,$$

in the local trivializations $P \cong M \times G$, $P' \cong M' \times G$ (cf. also [G]). From this, it follows that e.g. $F(p_0, e)$, where $e \in G$ denotes the identity, is completely determined by $j_{p_0}^1(f)$.

We now complete the proof of Theorem 4. Let $f^0, f^1: (M, p_0) \rightarrow (M', p'_0)$ be CR mappings as in the theorem. Denote by $F^0, F^1: P \rightarrow P'$ the lifts of f^0 and f^1 , respectively. As mentioned above, the assumption that $j_{p_0}^1(f^0) = j_{p_0}^1(f^1)$ implies that $F^0(p_0, e) = F^1(p_0, e)$. The conclusion of Theorem 4 is now a consequence of the following simple uniqueness result, whose proof is left to the reader.

Lemma 5.1.4. *Let $\eta_1, \dots, \eta_k$ and $\eta'_1, \dots, \eta'_k$ be 1-forms near the origin in $\mathbb{R}^k$ such that $\eta_1(0), \dots, \eta_k(0)$ as well as $\eta'_1(0), \dots, \eta'_k(0)$ span $T_0\mathbb{R}^k$. Assume that $f^1, f^2: (\mathbb{R}^k, 0) \rightarrow (\mathbb{R}^k, 0)$ are smooth mappings such that*

$$(5.1.5) \quad (f^i)^*(\eta'_j) = \eta_j, \quad i = 1, 2, \quad j = 1, \dots, k.$$

Then $f^1 \equiv f^2$ near 0.

This completes the proof of Theorem 4. $\square$

5.2. Proof of Theorem 5. In view of Theorem 4, it suffices to prove the last statement in Theorem 5, namely that f^0 is a diffeomorphism. To prove this, we shall apply results about geometric properties of CR mappings which impose conditions, not previously introduced in this paper, on the manifolds. We shall refer the reader to [BER] for relevant definitions and basic results.

For the remainder of this section, we shall denote f^0 simply by f . We first claim that the generic rank of f must in fact be 5 ($= \dim_{\mathbb{R}} M$). Indeed, since M is 2-nondegenerate at p_0 , there is no formal holomorphic vector field tangent to it at p_0 (see [BER, Theorem 11.7.5]). Moreover, since M is in addition minimal at p_0 , [BR3, Theorem 2] implies that the generic rank of f is either full (i.e. 5) or even, and hence, by the generic rank assumption on f , the generic rank is either 4 or 5. It is shown in [BR3] that if the generic rank of f is 4, then M' contains 2-dimensional complex submanifolds (i.e. complex hypersurfaces) through points $p' \in M'$ arbitrarily close to p'_0 . Hence, in this case M' would not be minimal in any neighborhood of p_0 . This contradicts the assumption that M' is 2-nondegenerate at p'_0 , since finite nondegeneracy implies minimality and finite nondegeneracy is an open condition. This proves the claim above.

We now choose local coordinates $Z = (z, w) \in \mathbb{C}^2 \times \mathbb{C}$ vanishing at p_0 , and $Z' = (z', w') \in \mathbb{C}^2 \times \mathbb{C}$ vanishing at p'_0 such that M and M' are defined, respectively, near $p_0 = (0, 0)$ and $p'_0 = (0, 0)$ by

$$(2.1) \quad \operatorname{Im} w = \phi(z, \bar{z}, \operatorname{Re} w), \quad \text{and} \quad \operatorname{Im} w' = \phi'(z', \bar{z}', \operatorname{Re} w')$$

where $\phi(0) = \phi'(0, 0) = 0$ and $d\phi(0) = d\phi'(0) = 0$. Let $\mathcal{H}(Z)$ denote the formal power series in $Z = (Z_1, Z_2, Z_3)$ associated with the smooth CR mapping f at p_0 (see [BER, Proposition 1.7.4]). To show that f is a local CR diffeomorphism at $p_0 = 0$ it suffices to show that the formal mapping $\mathcal{H}$ is invertible at 0. In the coordinates (z, w) , and (z', w') introduced above, we can write $\mathcal{H}(z, w) = (\mathcal{F}(z, w), \mathcal{G}(z, w))$, where $\mathcal{F}$ is valued in $\mathbb{C}^2$ and $\mathcal{G}$ is valued in $\mathbb{C}$. The fact that f maps M into M' implies that

$$(5.2.1) \quad \mathcal{G}(z, s + i\phi(z, \bar{z}, s)) \sim \phi'(\mathcal{F}(z, s + i\phi(z, \bar{z}, s)), \bar{\mathcal{F}}(\bar{z}, s - i\phi(z, \bar{z}, s)), \operatorname{Re} \mathcal{G}(z, s + i\phi(z, \bar{z}, s))),$$

where $\sim$ denotes equality of power series. By applying the two vector fields

$$(5.2.2) \quad L_j := \frac{\partial}{\partial \bar{z}_j} - 2i \frac{\phi_{\bar{z}_j}}{1 + i\phi_s} \frac{\partial}{\partial w}, \quad j = 1, 2,$$

which are tangent to M and form a basis for the CR vector fields near 0, to (5.2.1) and setting $(z, \bar{z}, s) = (0, 0, 0)$, we first conclude that $\partial \mathcal{G} / \partial z_j(0) = 0$ for $j = 1, 2$. We shall prove that $\mathcal{H}$ is invertible by showing that $\partial \mathcal{G} / \partial w(0) \neq 0$ and $\det(\partial \mathcal{F}_j / \partial z_k(0)) \neq 0$, where $j, k = 1, 2$.

Recall from the proof of Theorem 4 above that both M and M' are Levi uniform of rank 1 at p_0 and p'_0 , respectively. It follows that there is a smooth choice of Levi form in a neighborhood of p_0 in M which is positive semidefinite in that neighborhood and, hence, M is pseudoconvex near p_0 . A similar argument shows that M' is also pseudoconvex near p'_0 . Since M' is both minimal and pseudoconvex at $p'_0 = 0$, it is also minimally convex (see [BR4] for the definition of minimal convexity

and this particular statement). Since $f: (M, p_0) \rightarrow (M', p'_0)$, where M is minimal at p_0 and M' is minimally convex at p'_0 , has generic full rank, [BR4, Theorem 2] implies that $\partial\mathcal{G}/\partial w(0,0) \neq 0$. (This fact can also be deduced by combining [Be, Theorem 3] with [BER, Proposition 9.4.10].) To show that $\det(\partial\mathcal{F}_j/\partial z_k(0)) \neq 0$, where $j, k = 1, 2$, we point out that finite nondegeneracy implies essential finiteness. Indeed, finite nondegeneracy at a point is equivalent to essential finiteness and essential type 1 at that point (see [BER, Proposition 11.8.27]; the reader is also referred to [BER, Chapter XI] for the definitions of essential finiteness and essential typ). By [BR2, Theorem 2] and [BR1, Theorem 3], we have

$$(5.2.3) \quad \dim_{\mathbb{C}} \mathbb{C}[[z]]/\mathcal{I}(\mathcal{F}_1(z, 0), \mathcal{F}_2(z, 0)) = 1,$$

where $\mathbb{C}[[z]]$ denotes the ring of formal power series in $z = (z_1, z_2)$ and the notation $\mathcal{I}(\mathcal{F}_1(z, 0), \mathcal{F}_2(z, 0))$ stands for the ideal in $\mathbb{C}[[z]]$ generated by $\mathcal{F}_1(z, 0)$ and $\mathcal{F}_2(z, 0)$. It is not difficult to see that (5.2.3) is equivalent to $\det(\partial\mathcal{F}_j/\partial z_k(0)) \neq 0$. This completes the proof of the fact that f is a local CR diffeomorphism near p_0 . and, hence, that of Theorem 5. $\square$

5.3. An example showing necessity of the generic rank condition in

Theorem 5. Recall that $\Gamma_{\mathbb{C}} \subset \mathbb{C}^3$ denotes the smooth locus of the tube over the light cone in $\mathbb{R}^3$ (see Example 0.2.1 and §4). Observe that $\Gamma_{\mathbb{C}}$ is geometrically convex. Let us fix a point, say $p_0 = (1, 0, 1) \in \Gamma_{\mathbb{C}}$. The convexity of $\Gamma_{\mathbb{C}}$ near p_0 implies that the real linear form $\operatorname{Re} \ell(Z)$, where $\ell(Z) := \operatorname{Re} Z_3 - \operatorname{Re} Z_1$, is nonnegative for all $p \in \Gamma_{\mathbb{C}}$ near p_0 . (This is of course easy to check directly as well.) Define, for $Z \in \mathbb{C}^3$ in the half space $S := \{\operatorname{Re} \ell(Z) \geq 0\}$,

$$(5.3.1) \quad h(Z) := \sqrt{\ell(Z)},$$

where the square root in (5.3.1) is the branch in the right halfplane which preserves the point $\zeta = 1$. The reader can verify that the function

$$(5.3.2) \quad J(Z) := e^{-1/h(Z)}$$

is holomorphic in the open half plane D , C^∞ in its closure $\overline{D}$, and vanishes to infinite order on the boundary ∂D (see also [BER, Example 1.4.9]). Since $\Gamma_{\mathbb{C}}$ is locally contained in $\overline{D}$ near p_0 , meets D in a dense open subset near p_0 , and meets ∂D at p_0 (indeed, along a complex line contained in $\Gamma_{\mathbb{C}}$), we deduce that the restriction j of J to a neighborhood of p_0 in $\Gamma_{\mathbb{C}}$ is a smooth CR function with generic (real) rank 2 which vanishes to infinite order at p_0 .

Let $M' \subset \mathbb{C}^3$ be any smooth real hypersurface which is Levi uniform of rank 1 at $p'_0 \in M'$. This implies, in particular, that M' is foliated near p'_0 by complex curves (see e.g. Proposition 1.12). Let $\gamma: (\mathbb{C}, 0) \rightarrow (\mathbb{C}^3, p'_0)$ be a parametrization of the complex curve through p'_0 . The CR mapping $f^0: (\Gamma_{\mathbb{C}}, p_0) \rightarrow (M', p'_0)$ defined by $f^0(p) := \gamma(J(p))$, for $p \in \Gamma_{\mathbb{C}}$ near p_0 , is then smooth, of generic rank 2, and vanishes to infinite order at p_0 . Hence, if we also choose M to be 2-nondegenerate at p'_0 , all conditions of Theorem 5 except the generic rank condition on f^0 are satisfied, but the conclusion fails.

6. EXAMPLES OF EVERYWHERE LEVI DEGENERATE CR MANIFOLDS AND COMPUTATIONS OF $\hat{k}$

Trivial examples of real hypersurfaces in $\mathbb{C}^{n+1}$ which are everywhere Levi degenerate can be obtained by taking any hypersurface of the form $\tilde{M} \times \mathbb{C}$, where $\tilde{M}$ is a real hypersurface in $\mathbb{C}^n$. Such hypersurfaces, as mentioned in the introduction, are never 2-nondegenerate and, hence, are not of interest to us in the present paper. We shall give here two (from our viewpoint) more interesting situations where everywhere Levi degenerate hypersurfaces in $\mathbb{C}^{n+1}$ arise naturally. We shall also exhibit two classes of such manifolds for which the invariant $\hat{k}$ satisfies $\hat{k} \equiv 2i$ and, hence, for which Theorem 2 is applicable; see Propositions 6.13 and 6.27 below. We shall conclude with a result showing that the curvature in Theorem 2 is a nontrivial invariant.

Example 6.1 (Everywhere characteristic hypersurfaces). Let $p(Z, \partial) := p(Z, \partial/\partial Z)$ be a homogeneous holomorphic partial differential operator, i.e. $p(Z, \zeta)$ is a holomorphic function of (Z, ζ) in some domain $U \times \mathbb{C}^{n+1} \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ and a homogeneous polynomial in ζ , and

$$(6.2) \quad \partial/\partial Z = (\partial/\partial Z_1, \dots, \partial/\partial Z_{n+1}).$$

($p(Z, \zeta)$ is called the principal symbol of the operator $p(Z, \partial)$.) A real hypersurface $M \subset \mathbb{C}^{n+1}$ is called *characteristic* at $p_0 \in M \cap U$ for $p(Z, \partial)$ (or, more generally, any holomorphic partial differential operator whose principal symbol is $p(Z, \zeta)$) if

$$(6.3) \quad p(p_0, \partial\rho(p_0, \bar{p}_0)/\partial Z) = 0,$$

where $\rho(Z, \bar{Z}) = 0$ is a defining equation for M near p_0 , and M is called *everywhere characteristic* if $M \cap U$ is characteristic at every point. Observe that if $p(Z, \partial)$ is homogeneous of degree one (i.e. it is a holomorphic vector field), then M is everywhere characteristic for $p(Z, \partial)$ if and only if the holomorphic vector field $p(Z, \partial)$ is tangent to M in U . Hence, being everywhere characteristic for a holomorphic partial differential operator can be viewed as a higher degree analogue of holomorphic degeneracy (see [BER, Chapter XI]).

Everywhere characteristic hypersurfaces, relative to a given operator $p(Z, \partial)$, arise as natural boundaries for the holomorphic continuation of (holomorphic) solutions of $p(Z, \partial)u = 0$ (see e.g. [Hö, Chapter IX.4]). A concrete example is given by the so-called *Lie ball* defined by the equation

$$(6.4) \quad |Z|^2 + \left(|Z|^4 - \left| \sum_{k=1}^{n+1} Z_k^2 \right|^2 \right)^{1/2} < 1.$$

The Lie ball is the maximal domain in $\mathbb{C}^{n+1}$ to which every harmonic function in the unit ball of $\mathbb{R}^{n+1}$ can be holomorphically continued (see e.g. [A]; cf. also [E1]). The boundary of the Lie ball is everywhere characteristic (at every smooth point) for the “Laplace operator”

$$(6.5) \quad \sum_{j=1}^{n+1} (\partial/\partial Z_j)^2.$$

Another example is the tube over the light cone (see Example 0.2.1) which is everywhere characteristic for the “wave operator”

$$(6.6) \quad \sum_{j=1}^n (\partial/\partial Z_j)^2 - (\partial/\partial Z_{n+1})^2.$$

We have the following.

Proposition 6.7. *Let $M \subset \mathbb{C}^{n+1}$ be a real hypersurface which is everywhere characteristic for a homogeneous partial differential operator $p(Z, \partial)$. Then M is everywhere Levi degenerate.*

Proof. Pick $p_0 \in M$ and let $\rho(Z, \bar{Z}) = 0$ be a defining equation for M near $p_0 \in M$. We first claim that the CR vector field

$$(6.8) \quad L := \sum_{k=1}^{n+1} \overline{p_{\zeta_k}(Z, \rho_Z(Z, \bar{Z}))} \frac{\partial}{\partial \bar{Z}_k},$$

where $\rho_Z := \partial\rho/\partial Z$ and $p_\zeta = \partial p/\partial \zeta$, is tangent to M . Indeed, since M is everywhere characteristic for $p(\partial)$, we have

$$(6.9) \quad p(Z, \rho_Z(Z, \bar{Z})) = a(Z, \bar{Z})\rho(Z, \bar{Z})$$

for some function a . The claim now follows from Euler’s formula. By differentiating (6.9) with respect to $\bar{Z}$, it is straightforward (and left to the reader) to verify that L is a nullvector for the Levi form at every $p \in M$ near p_0 . This proves the proposition. $\square$

Example 6.10 (Tubes over homogeneous algebraic varieties). Let $p(x)$ be a homogeneous polynomial in $x = (x_1, \dots, x_{n+1})$ with real coefficients and assume that $\partial p/\partial x$ is not identically zero along the variety $V_{\mathbb{R}} := \{x \in \mathbb{R}^{n+1} : p(x) = 0\}$. Then the tube $V_{\mathbb{C}}$ over $V_{\mathbb{R}}$ in $\mathbb{C}^{n+1}$,

$$(6.11) \quad V_{\mathbb{C}} := \{Z \in \mathbb{C}^{n+1} : p(\operatorname{Re} Z) = 0\},$$

is a real hypersurface (outside a lower dimensional real algebraic subvariety) which we denote by M . The “radial” CR vector field

$$(6.12) \quad L = \sum_{j=1}^{n+1} \operatorname{Re} Z_j \partial/\partial \bar{Z}_j$$

is tangent to M , and the reader can easily verify that L is a null vector for the Levi form of M at every $p \in M$. A concrete example is again the tube over the light cone (Example 0.2.1). Another example is the cubic defined by (6.11) with $p(x) = x_1^3 + x_2^3 - x_3^3$ which was given by Freeman [F1] as an example of a manifold foliated by complex curves but not locally biholomorphic to a manifold of the form $\tilde{M} \times \mathbb{C}$.

We shall compute the invariant $\hat{k}$ for two classes of real hypersurface $M \subset \mathbb{C}^3$ that share some interesting properties with the tube over the light cone, and show that Theorem 2 is applicable for manifolds from these classes. First, we shall consider hypersurfaces $M \subset \mathbb{C}^3$ which are tubes over homogeneous algebraic varieties in $\mathbb{R}^3$ as in Example 6.10 above. For convenience, we shall replace $\operatorname{Re} Z$ above by $Z + \bar{Z}$.

Proposition 6.13. *Let $M \subset \mathbb{C}^3$ be a real hypersurface which is contained in the real algebraic variety*

$$(6.14) \quad V := \{Z \in \mathbb{C}^3 : p(Z + \bar{Z}) = 0\},$$

where $p(x)$ is a real-valued homogeneous polynomial. Assume that M is Levi uniform and 2-nondegenerate at $Z_0 \in M$. Then, the invariant $\hat{k} \equiv 2i$ in a neighborhood of Z_0 .

Proof. For simplicity, we shall write $x = Z + \bar{Z}$. There is no loss of generality in assuming that $x_1^0 \neq 0$ and $p_{x_3}(x^0) \neq 0$. As a basis for the CR vector fields on M near Z_0 we choose

$$(6.15) \quad L_1 = \frac{1}{N_1}(p_{x_3}\partial/\partial Z_2 - p_{x_2}\partial/\partial Z_3), \quad L_2 = \frac{1}{N_2}\sum_{j=1}^3 x_j\partial/\partial Z_j,$$

where N_1, N_2 are factors to be chosen. As mentioned above, L_2 spans the Levi null bundle $\mathfrak{N}$ near Z_0 . For $\theta, \theta^1, \theta^2$, we choose (the pullbacks to M of)

$$(6.16) \quad \theta = i \sum_{j=1}^3 p_{x_j} dZ_j,$$

and

$$(6.17) \quad \theta^1 = \frac{N_1}{x_1 p_{x_3}}(x_1 dZ_2 - x_2 dZ_1), \quad \theta^2 = \frac{N_2}{x_1} dZ_1.$$

The reader can verify that θ spans $T^0 M$, θ, θ^1 span $T'' M$, and $\theta, \theta^1, \theta^2$ span $T' M$ as defined in section 2. Moreover, we have $\langle \theta^\alpha, L_\beta \rangle = \delta_\beta^\alpha$, and a calculation shows that the non-zero entry $g_{\bar{1}1}$ of the Levi form is given by

$$(6.18) \quad g_{\bar{1}1} := \frac{1}{i} \langle d\theta, L_{\bar{1}} \wedge L_1 \rangle = \frac{1}{N_1^2} Q(x),$$

where $Q(x)$ denotes the quadratic form

$$(6.19) \quad Q(x) = p_{x_3}^2 p_{x_2 x_2} - 2p_{x_2} p_{x_3} p_{x_2 x_3} + p_{x_2}^2 p_{x_3 x_3}.$$

Since M is assumed Levi uniform and 2 nondegenerate (and hence Levi uniform of rank 1; see the proof of Theorem 5), we have $Q(x) \neq 0$ near x^0 . We may assume that $Q(x) > 0$ in a neighborhood of x^0 , since otherwise we could replace $p(x)$ by $-p(x)$. So, we choose

$$(6.20) \quad N_1(x) = \sqrt{Q(x)}$$

and conclude that $g_{\bar{1}1} \equiv 1$ near x^0 .

To compute the only non-zero element $h_{\bar{1}\bar{1}2}$ in the third order tensor, we use Lemma 2.12, equation (2.16), and the fact that $g_{\bar{1}1} = 1$. After a straightforward calculation, we obtain,

$$(6.21) \quad 2ih_{\bar{1}\bar{1}2} = \langle d\theta^1, L_{\bar{1}} \wedge L_2 \rangle = \frac{p_{x_3}}{N_2 \sqrt{Q}} \left(x_2 \frac{\partial}{\partial x_2} \left(\frac{\sqrt{Q}}{p_{x_3}} \right) - x_1 \frac{\partial}{\partial x_2} \left(\frac{x_2 \sqrt{Q}}{x_1 p_{x_3}} \right) \right) - \frac{p_{x_2}}{N_2 \sqrt{Q}} \left(x_2 \frac{\partial}{\partial x_3} \left(\frac{\sqrt{Q}}{p_{x_3}} \right) - x_1 \frac{\partial}{\partial x_3} \left(\frac{x_2 \sqrt{Q}}{x_1 p_{x_3}} \right) \right).$$

Let us simplify the expression

$$(6.22) \quad E(x) := \frac{p_{x_3}}{\sqrt{Q}} \left(x_2 \frac{\partial}{\partial x_2} \left(\frac{\sqrt{Q}}{p_{x_3}} \right) - x_1 \frac{\partial}{\partial x_2} \left(\frac{x_2 \sqrt{Q}}{x_1 p_{x_3}} \right) \right) - \frac{p_{x_2}}{\sqrt{Q}} \left(x_2 \frac{\partial}{\partial x_3} \left(\frac{\sqrt{Q}}{p_{x_3}} \right) - x_1 \frac{\partial}{\partial x_3} \left(\frac{x_2 \sqrt{Q}}{x_1 p_{x_3}} \right) \right)$$

appearing in (6.21). Observe that, for $j = 2, 3$,

$$(6.23) \quad \frac{\partial}{\partial x_j} \left(\frac{x_2 \sqrt{Q}}{x_1 p_{x_3}} \right) = \delta_{2j} \frac{1}{x_1} \frac{\sqrt{Q}}{p_{x_3}} + \frac{x_2}{x_1} \frac{\partial}{\partial x_j} \left(\frac{\sqrt{Q}}{p_{x_3}} \right).$$

Substituting this in (6.22), we find

$$(6.24) \quad E(x) = 1.$$

Thus, (6.21) can be simplified to $2ih_{\bar{1}\bar{1}2} = 1/N_2$ and, hence, we choose

$$(6.25) \quad N_2(x) = \frac{1}{2i}$$

so that $h_{\bar{1}\bar{1}2} = 1$. With these choices of N_1, N_2 , as is shown in section 3, the invariant $\hat{k}$ is computed from the formula

$$(6.26) \quad \begin{aligned} \hat{k} &= \langle d\theta^2, L_{\bar{2}} \wedge L_2 \rangle \\ &= -\frac{N_2}{x_1^2} \frac{1}{|N_2|^2} \left\langle d\bar{Z}_1 \wedge dZ_1, \sum_{j,k=1}^3 x_j x_k \partial/\partial \bar{Z}_j \wedge \partial/\partial Z_k \right\rangle \\ &= -\frac{1}{\bar{N}_2} = 2i. \end{aligned}$$

This completes the proof of Proposition 6.13. $\square$

We shall also compute $\hat{k}$ for tubes over hypersurfaces in $\mathbb{R}^3$ which are everywhere characteristic for the wave operator. Observe that if $\{x \in \mathbb{R}^3 : f(x) = 0\}$ is a hypersurface in $\mathbb{R}^3$ which is everywhere characteristic for the wave operator $(\partial/\partial x_1)^2 + (\partial/\partial x_2)^2 - (\partial/\partial x_3)^2$, then the real hypersurface $M \subset \mathbb{C}^3$ defined by $M = \{Z \in \mathbb{C}^3 : f(\operatorname{Re} Z) = 0\}$ is everywhere characteristic for the complex wave operator $(\partial/\partial Z_1)^2 + (\partial/\partial Z_2)^2 - (\partial/\partial Z_3)^2$ and, hence, of the type considered in Example 6.1. As above, we shall use $Z + \bar{Z}$ instead of $\operatorname{Re} Z$.

Proposition 6.27. *Let $M \subset \mathbb{C}^3$ be a real smooth hypersurface, defined locally near $Z^0 \in M$ by an equation of the form $f(Z + \bar{Z}) = 0$. Assume that the hypersurface $\{f(x) = 0\} \subset \mathbb{R}^3$ is everywhere characteristic for the wave operator near $x^0 = Z^0 + \bar{Z}^0$, i.e.*

$$(6.28) \quad f_{x_1}^2 + f_{x_2}^2 - f_{x_3}^2 = 0,$$

and that M is Levi uniform and 2-nondegenerate at Z^0 . Then, $\hat{k} \equiv 2i$ near Z^0 .

Proof. As in the proof of Proposition 6.13, we shall use $x = \bar{Z} + \bar{Z}$. We observe that we may choose

$$(6.29) \quad L_2 = \frac{1}{N_2} (f_{x_1} \partial/\partial Z_1 + f_{x_2} \partial/\partial Z_2 - f_{x_3} \partial/\partial Z_3)$$

as a vector field spanning the Levi nullbundle $\mathfrak{N}$ near Z^0 . Also, take

$$(6.30) \quad \theta^1 = \frac{N_1}{f_{x_1}f_{x_3}} (f_{x_1}dZ_2 - f_{x_2}dZ_1)$$

and choose L_1 , θ , and θ^2 as in the proof of Proposition 6.13 (replacing p by f , of course). By repeating the calculations carried out in the proof of Proposition 6.13 and choosing N_1 , N_2 accordingly, we deduce that if $f_{x_1}(x^0)f_{x_3}(x^0) \neq 0$ (which we may assume without loss of generality), then

$$(6.31) \quad \hat{k}(Z) = -2i \frac{f_{x_1}(x)}{R(x)^2} \left(X \left(\frac{R}{f_{x_1}} \right) \right) (x),$$

where

$$(6.32) \quad R = \frac{f_{x_2}f_{x_1}}{f_{x_3}} \frac{\partial}{\partial x_3} \left(\frac{f_{x_2}}{f_{x_1}} \right) - f_{x_1} \frac{\partial}{\partial x_2} \left(\frac{f_{x_2}}{f_{x_1}} \right)$$

and

$$(6.33) \quad X = f_{x_1} \frac{\partial}{\partial x_1} + f_{x_2} \frac{\partial}{\partial x_2} - f_{x_3} \frac{\partial}{\partial x_3}.$$

The details of these calculations are left to the reader. Pick $Z^1 \in M$ near Z^0 . After a translation and applying a real Lorentz transformation if necessary, we may assume that $Z^1 = 0$, $-f_{x_1}(0) = f_{x_3}(0) = 1$, and $f_{x_2}(0) = 0$. By solving for x_3 in the equation $f(x) = 0$, we may also assume that $f(x) = x_3 - \phi(x_1, x_2)$, where $\phi(x_1, x_2)$ satisfies $\phi(0, 0) = \phi_{x_2}(0, 0) = 0$, $\phi_{x_1}(0, 0) = 1$, and

$$(6.34) \quad \phi_{x_1}^2 + \phi_{x_2}^2 = 1.$$

A straightforward calculation shows that M is Levi uniform of rank one and 2-nondegenerate at 0 if and only if $\phi_{x_2x_2}(0, 0) \neq 0$. By differentiating (6.34), we find that

$$(6.35) \quad \phi_{x_1x_j}(0, 0) = 0, \quad \phi_{x_1x_jx_k}(0, 0) = -\phi_{x_2x_j}(0, 0)\phi_{x_2x_k}(0, 0),$$

for $j, k = 1, 2$. By substituting $f(x) = x_3 - \phi(x_1, x_2)$ in (6.31) and using (6.35), we conclude $\hat{k}(Z^1) = 2i$. The proof of Proposition 6.27 is now complete since Z^1 was arbitrary. $\square$

We shall conclude the paper with the following result which implies that in general the curvature in Theorem 2 is nontrivial.

Proposition 6.36. *For an integer $k \geq 2$, let*

$$(6.37) \quad V_k := \{Z \in \mathbb{C}^3 : (Z_1 + \bar{Z}_1)^k + (Z_2 + \bar{Z}_2)^k - (Z_3 + \bar{Z}_3)^k = 0\}.$$

If $k \geq 3$, then, for any $p \in V_k$, there is no local biholomorphism near p sending V_k into V_2 (which is the tube over the light cone).

proof. Suppose, in order to reach a contradiction, that there exists a $p \in V_k$ and a local biholomorphism $Z' = H(Z)$ near p with $H(V_k) \subset V_2$. There is no loss of

generality in assuming that p is a smooth point of $V_k \cap \mathbb{R}^3$, and that $H(p)$ is a smooth point of V_2 . Since V_2 is everywhere characteristic for the complex wave operator (6.6), V_k must be everywhere characteristic for the homogeneous second order partial differential operator with principal symbol

$$(6.38) \quad p(Z; \zeta) = \sum_{i,j=1}^3 \left(\frac{\partial Z'_i}{\partial Z_1} \frac{\partial Z'_j}{\partial Z_1} + \frac{\partial Z'_i}{\partial Z_2} \frac{\partial Z'_j}{\partial Z_2} - \frac{\partial Z'_i}{\partial Z_3} \frac{\partial Z'_j}{\partial Z_3} \right) \zeta_i \zeta_j$$

(cf. e.g. [Hö, Chapter VI]); i.e. we must have

$$(6.39) \quad p(Z; (x_1^{k-1}, x_2^{k-1}, -x_3^{k-1})) = a(Z, \bar{Z})(x_1^k + x_2^k - x_3^k),$$

where, as above, $x = Z + \bar{Z}$. By restricting to real values of Z and identifying the homogeneous terms of order $2k - 2$, we get

$$(6.40) \quad Q(x_1^{k-1}, x_2^{k-1}, -x_3^{k-1}) = b(x)(x_1^k + x_2^k - x_3^k),$$

for some nondegenerate quadratic form $Q(\zeta)$ and some homogeneous polynomial $b(x)$ of degree $k - 2$. It is not difficult to see, and left to the reader to verify, that this cannot happen unless $k = 2$. This completes the proof of Proposition 6.36. $\square$

REFERENCES

- [A] V. Avanissian, *Cellule d'Harmonicité et Prolongement Analytique Complexe*, Travaux en Cours, Hermann, Paris, 1985.
- [BER] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild, *Real Submanifolds In Complex Space and Their Mappings*, Princeton Mathematical Series 47, Princeton University Press, Princeton, NJ, 1999.
- [BHR] M. S. Baouendi, X. Huang and L. P. Rothschild, *Regularity of CR mappings between algebraic hypersurfaces*, Invent. Math. **125** (1996), 13–36.
- [BJT] M. S. Baouendi, H. Jacobowitz and F. Treves, *On the analyticity of CR mappings*, Ann. Math. **122** (1985), 365–400.
- [BR1] M. S. Baouendi and L. P. Rothschild, *Germes of CR maps between real analytic hypersurfaces*, Invent. Math. **93** (1988), 481–500.
- [BR2] ———, *Geometric properties of mappings between hypersurfaces in complex space*, J. Differential Geom. **31** (1990), 473–499.
- [BR3] ———, *Remarks on the generic rank of a CR mapping*, J. Geom. Anal. **2** (1992), 1–9.
- [BR4] ———, *A generalized complex Hopf Lemma and its application to CR mappings*, Invent. Math. **111** (1993), 331–148.
- [Be] S. Bell, *Local regularity of CR homeomorphisms*, Duke Math. J. **57** (1988), 295–300.
- [BS] D. Burns, Jr. and S. Shnider, *Real hypersurfaces in complex manifolds*, Proceedings of Symposia in Pure Mathematics XXX, Part 2, Several Complex Variables, Amer Math. Soc., Providence, RI, 1977, pp. 141–168.
- [CS] A. Čap, H. Schichl, *Parabolic geometries and canonical Cartan connections* (1997), (preprint).
- [C1] E. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, I*, Ann. Math. Pura Appl. **11** (1932), 17–90, (or Oeuvres II, 1231–1304).
- [C2] ———, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, II*, Ann. Scuola Norm. Sup. Pisa **1** (1932), 333–354, (or Oeuvres III, 1217–1238).
- [CM] S.-S. Chern and J.K. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.
- [E1] P. Ebenfelt, *Holomorphic extension of solutions of elliptic partial differential equations and a complex Huygens' principle*, J. London Math. Soc. **55** (1997), 87–104.

- [E2] ———, *Nondegeneracy conditions and normal forms for real hypersurfaces in complex space*, Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1997), École Polytech., Palaiseau.
- [E3] ———, *Normal forms and the biholomorphic equivalence problem for real hypersurfaces in $\mathbb{C}^3$* , Indiana Univ. Math. J. **47** (1998), 311–366.
- [E4] ———, *New invariant tensors in CR structures and a normal form for real hypersurfaces at a generic Levi degeneracy*, J. Differential Geom. **50** (1998), 207–247.
- [Er] A. Ershova, *Automorphisms of 2-nondegenerate hypersurfaces in $\mathbb{C}^3$* , (preprint; 1999).
- [EIS] V. V. Ezhov, A. V. Isaev, G. Schmalz, *Invariants of elliptic and hyperbolic CR structures of codimension 2*, Int. J. Math. **10** (1999), 1–52.
- [F1] M. Freeman, *Real submanifolds with degenerate Levi form*, Proceedings of Symposia in Pure Mathematics XXX, part I, Several Complex Variables, Amer. Math. Soc., Providence, RI, 1977.
- [F2] ———, *Local biholomorphic straightening of real submanifolds*, Ann. of Math. **106** (1977), 319–352.
- [G] R. B. Gardner, *The Method of Equivalence and Its Applications*, SIAM CBMS-NSF Regional Conference Series in Applied Mathematics, Philadelphia, 1989.
- [GM] T. Garrity, R. Mizner, *The equivalence problem for higher-codimensional CR structures*, Pacific J. Math. **177** (1997), 211–235.
- [Ha] C.-K. Han, *Complete differential system for the mappings of CR manifolds of nondegenerate Levi forms*, Math. Ann. **309** (1997), 401–409.
- [Hay] A. Hayashimoto, *On the complete system of finite order for CR mappings and its application*, Osaka J. Math. **35** (1998), 617–628.
- [He] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York and London, 1962.
- [Hö] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin, 1983.
- [K] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer-Verlag, Berlin, 1972.
- [KN] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry, Volume I and II*, Interscience Publ., New York, 1963.
- [SS] G. Schmalz, J. Slovák, *The geometry of hyperbolic and elliptic CR manifolds of codimension two* (1999), (preprint).
- [SV] A. G. Sergeev, V. S. Vladimirov, *Complex analysis in the future tube*, Several Complex Variables II, Encyclopaedia of Mathematical Sciences, Vol. 8, Springer-Verlag, Berlin, 1994.
- [Si] C. L. Siegel, *Symplectic geometry*, Amer. J. Math. **65** (1943), 1–86.
- [Sta1] N. Stanton, *Infinitesimal CR automorphisms of rigid hypersurfaces*, Amer. J. Math. **117** (1995), 141–167.
- [Sta2] ———, *Infinitesimal CR automorphisms*, Amer. J. Math. **118** (1996), 209–233.
- [Ste] S. Sternberg, *Lectures on Differential Geometry*, Second edition, Chelsea Publ. Co., New York, 1983.
- [T1] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables*, J. Math. Soc. Japan **14** (1962), 397–429.
- [T2] ———, *On generalized graded Lie algebras and geometric structures. I*, J. Math. Soc. Japan **19** (1967), 215–254, (erratum **36**, p. 1568).
- [We1] S. M. Webster, *Pseudohermitian structures on a real hypersurface*, J. Differential Geom. **13** (1978), 25–41.
- [We2] ———, *The holomorphic contact geometry of a real hypersurface*, Modern Methods in Complex Analysis (T. Bloom et al, eds.), Annals of Mathematics Studies 137, Princeton University Press, Princeton, N.J., 1995, pp. 327–342.
- [Wo] P. P. Wong, *A construction of normal forms for weakly pseudoconvex CR manifolds in $\mathbb{C}^2$* , Invent. Math. **69** (1982), 311–329.

DEPARTMENT OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY, 100 44 STOCKHOLM, SWEDEN

E-mail address: ebenfelt@math.kth.se