Math 289A, Spring 2013

Markov Chain Limit Theorem

1. Let $X = (X_t : t \geq 0)$ be an irreducible continuous time Markov chain with finite state space E. Other notation will be that used in class.

2. Invariant distribution. Recall from class that if we fix a state $b \in E$ then

$$\nu_b(j) := \mathbf{E}^b \int_0^{R_b} 1_{\{X_s = j\}}\, ds, \qquad j \in E, \tag{2.1}$$

defines an invariant measure ν_b on E (that is, $\nu_b P_t = \nu_b$ for all $t > 0$) with total mass

$$\sum_{j \in E} \nu(i) = \mathbf{E}^b[R_b] < \infty. \tag{2.2}$$

If we normalize ν_b to obtain a probability distribution, then we must obtain m, since m is the unique stationary distribution for the Markov chain. That is,

$$\frac{\nu_b(j)}{\mathbf{E}^b[R_b]} = m(j), \qquad \forall j \in E, \forall b \in E. \tag{2.3}$$

Observe that

$$\nu_b(b) = \frac{1}{q(b)} \tag{2.4}$$

(the mean of the initial exponential holding time in state b under $\mathbf{P}^b$) Combining (2.4) with the special case of (2.3) in which $b = j$ we arrive at

$$m(j) = \frac{1}{q(j) \cdot \mathbf{E}^j[R_j]}. \tag{2.5}$$

3. Tauberian theorem. We shall make use of the following theorem. Let $F : [0, \infty) \to \mathbf{R}$ be a continuously differentiable function with derivative f, and assume that $F(0) = 0$. We suppose that the Laplace transform integral

$$\varphi(\alpha) := \int_0^\infty e^{-\alpha t} f(t)\, dt, \tag{3.1}$$

is absolutely convergent for all $\alpha > 0$. Then

$$\lim_{t \to \infty} F(t) = L \tag{3.2}$$

if and only if

$$\lim_{\alpha \to 0} \varphi(\alpha) = L. \tag{3.3}$$

(This "only if" assertion follows easily from the formula

$$\varphi(\alpha) = \int_0^\infty e^{-u} F(u/\alpha)\, du,$$

which results from the definition of $\varphi(\alpha)$ after integration by parts. The "if" assertion is more difficult; see Theorem 2 on p. 445 of W. Feller's *Introduction to Probability Theory and its Applications, Vol. II* (Second edition).

4. A limit. We apply the result of **3.** to the function $F(t) = P_t(j,j) - 1$, which has continuous derivative $f(t) = QP_t(j,j)$. Observe that

$$|f(t)| \le CP_t(i,j), \tag{4.1}$$

where $C := \max_{i,j \in E} |Q(i,j)|$. Consequently the Laplace transform of f is finite for $\alpha > 0$, and we have

$$\varphi(\alpha) = \int_0^\infty e^{-\alpha t} f(t)\, dt = QU^\alpha(j,j) = \alpha U^\alpha(i,j) - 1, \qquad \alpha > 0. \tag{4.2}$$

We deduce from this and **3.** that

$$\lim_{t \to \infty} P_t(j,j) = \lim_{\alpha \to 0} \alpha U^\alpha(j,j), \tag{4.3}$$

the existence of the limit on the left side of (4.3) being part of the assertion, once we prove that the limit on the right side of (4.3) exists.

5. We now evaluate the limit on the right side of (4.3). If the Markov chain starts in state j, then there are two ways it can be in state j at time t. Either it hasn't yet left state j by time t, or it has left and returned. These two possibilities account for the two terms on the right side of the following identity (the strong Markov property at time R_j was used for the second):

$$P_t(j,j) = e^{-q(j)t} + \int_0^t \mathbf{P}^j[R_j \in ds] P_{t-s}(j,j)\, ds. \tag{5.1}$$

Taking Laplace transforms we find that

$$U^\alpha(j,j) = \frac{1}{\alpha + q(j)} + \mathbf{E}^j[e^{-\alpha R_j}] \cdot U^\alpha(j,j), \qquad \alpha > 0. \tag{5.2}$$

Solving for $U^\alpha(j,j)$ and then multiplying by α:

$$\alpha U^\alpha(j,j) = \frac{1}{\alpha + q(j)} \cdot \frac{\alpha}{1 - \mathbf{E}^j[e^{-\alpha R_j}]}. \tag{5.3}$$

But (think l'Hôpital's Rule)

$$\lim_{\alpha \to 0+} \frac{1 - \mathbf{E}^j[e^{-\alpha R_j}]}{\alpha} = \mathbf{E}^j[R_j]. \tag{5.4}$$

We conclude that

$$\lim_{\alpha \to 0+} \alpha U^\alpha(j,j) = \frac{1}{q(j) \cdot \mathbf{E}^j[R_j]}. \tag{5.5}$$

Combining this with (4.3) and (2.5) we finally arrivee at

$$\lim_{t \to \infty} P_t(j,j) = m(j), \qquad \forall j \in E, \tag{5.6}$$

where m is the unique stationary distribution for the Markov chain. As noted in class, the "off diagonal" case then follows easily:

$$\lim_{t \to \infty} P_t(i,j) = m(j), \qquad \forall i, j \in E, \tag{5.7}$$