1. [10 points]. Consider a birth-and-death process with state space \( \{0, 1, 2\} \) and rates \( \lambda_0 = 1, \lambda_1 = 2, \mu_1 = 2, \mu_2 = 3 \). As usual, \( T_0 = \min\{t : X(t) = 0\} \) is the hitting time of state 0. Use “first-jump analysis” to find \( v_i = \mathbb{E}[T_0 | X(0) = i] \) for \( i = 1, 2 \). (Clearly \( v_0 = 0 \).)

Solution. Started in state 2, the Markov chain holds there for an exponential time (rate 3) and then jumps to state 1. Therefore

\[
v_2 = \frac{1}{3} + v_1.
\]

Started in state 1, the Markov chain holds there for an exponential time (rate 4=2+2) and then jumps to state 0 with probability \( \frac{2}{2+2} = \frac{1}{2} \) or to state 2 with probability \( \frac{1}{2} \).

Therefore

\[
v_1 = \frac{1}{4} + \frac{1}{2}v_0 + \frac{1}{2}v_2 = \frac{1}{4} + \frac{1}{2}v_2.
\]

Adding these two equations (and canceling \( v_1 \) from both sides) we see that

\[
v_2 = \frac{7}{12} + \frac{1}{2}v_2,
\]

so that

\[
\frac{1}{2}v_2 = \frac{7}{12},
\]

and finally

\[
v_2 = \frac{7}{6}.
\]

Feeding this information into the first equation displayed above we find that

\[
v_1 = v_2 - \frac{1}{3} = \frac{5}{6}.
\]

2. [15 points]. A birth-and-death process \( X(t) \) has state space \( \{0, 1, 2, 3\} \), birth rates \( \lambda_k = 2 \) for \( k = 0, 1, 2 \), and death rates \( \mu_k = 3 \) for \( k = 1, 2, 3 \) (\( \mu_0 = \lambda_3 = 0 \)).

(a) Find the limit distribution \( \pi_j = \lim_{t \to \infty} P_{ij}(t) \), \( i, j = 0, 1, 2, 3 \), for this process.

(b) Suppose that \( \mathbb{P}[X(0) = i] = \pi_i \) for \( i = 0, 1, 2, 3 \). Find \( \mathbb{E}[X(3)] \).

Solution. (a) Evidently,

\[
\theta_k = (2/3)^k, \quad k = 0, 1, 2, 3.
\]
The sums of these is $65/27$, and so the limit distribution is given by
\[ \pi_0 = \frac{27}{65}, \quad \pi_1 = \frac{18}{65}, \quad \pi_2 = \frac{12}{65}, \quad \pi_3 = \frac{8}{65}. \]

(b) Because $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ is the stationary distribution, we have
\[ \pi P(t) = \pi, \quad \forall t > 0, \]
and so if the distribution of $X(0)$ is $\pi$ (as we are supposing) then the distribution of $X(3)$ is also $\pi$. Therefore
\[ \mathbb{E}[X(3)] = 0 \cdot \pi_0 + 1 \cdot \pi_1 + 2 \cdot \pi_2 + 3 \cdot \pi_3 = \frac{0 + 18 + 24 + 24}{65} = \frac{66}{65} = 1.015. \]

3. [15 points]. An idealized factory has an unlimited number of machines and a single repairman. [Note: Although the number of machines is unlimited, at any given time all but finitely many are “broken” and awaiting repair.] The time-until-failure of a machine is an exponentially distributed random variable with parameter 1. The repair time of a machine is an exponentially distributed random variable with parameter 2. The failures of the different machines are independent, and at most one machine is under repair at any time. We model the number of operating machines as a birth-and-death process $X(t)$ with state space $\{0, 1, 2, \ldots\}$.

(a) Find the birth rates $\lambda_n$ and the death rates $\mu_n$ for $X(t)$.

(b) Find the stationary distribution for this birth-and-death process.

(c) In the long run, what fraction of the time is the repairman idle?

Solution. (a) $\mu_n = n$ for all $n \geq 1$—each operating machine fails at rate 1, so the combined failure rate is $n$ when $n$ machines are in working order; and $\lambda_n = 2$ for all $n \geq 0$.

(b) In view of part (a),
\[ \theta_k = \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} = \frac{2^k}{k!}, \quad k \geq 0. \]

By the well-known exponential series
\[ \sum_{k=0}^{\infty} \theta_k = e^2, \]
and so the stationary distribution $\pi$ is given by the formula
\[ \pi_k = e^{-2} \frac{2^k}{k!}, \quad k \geq 0. \]
In other words, the stationary distribution is the Poisson distribution with mean 2.

(c) As the repairman is always working, this probability is 0.

4. **[20 points]**. A Markov chain \(X(t), t \geq 0,\) has state space \(\{0,1\}\) and transition probability matrix

\[
P(t) = \begin{bmatrix}
.2 + .8e^{-5t} & .8 - .8e^{-5t} \\
.2 - .2e^{-5t} & .8 + .2e^{-5t}
\end{bmatrix}.
\]

(a) Compute the infinitesimal matrix \(A = P'(0)\).

(b) Using the above formula for \(P(t)\), show that \(P'(t) = AP(t)\) for all \(t > 0\).

(c) Let \(\pi = (\pi_0, \pi_1)\) be the top row of the limit matrix \(\lim_{t \to \infty} P(t)\). Compute \(\pi A\).

**Solution.**

(a) Differentiating \(P(t)\) entry by entry, we see that

\[
P'(t) = \begin{bmatrix}
-4e^{-5t} & 4e^{-5t} \\
e^{-5t} & -e^{-5t}
\end{bmatrix},
\]

and in particular that

\[
A = P'(0) = \begin{bmatrix}
-4 & 4 \\
1 & -1
\end{bmatrix}.
\]

(b) The product of \(A\) and \(P(t)\) is

\[
A \cdot P(t) = \begin{bmatrix}
-8 - 3.2e^{-5t} + .8 - .8e^{-5t} & -3.2 + 3.2e^{-5t} + 3.2 + .8e^{-5t} \\
.2 + .8e^{-5t} - .2 + .2e^{-5t} & .8 - .8e^{-5t} - .8 - .2e^{-5t}
\end{bmatrix}
\]

which simplifies to

\[
\begin{bmatrix}
-4e^{-5t} & 4e^{-5t} \\
e^{-5t} & -e^{-5t}
\end{bmatrix},
\]

which is the same as \(P'(t)\) computed already in part (a). Done.

(c) Evidently

\[
\lim_{t \to \infty} P(t) = \begin{bmatrix}
.2 & .8 \\
.2 & .8
\end{bmatrix},
\]

and so \(\pi = [.2 \quad .8]\). Therefore

\[
\pi A = [.2 \quad .8] \cdot \begin{bmatrix}
-4 & 4 \\
1 & -1
\end{bmatrix} = [-.8 + .8 \quad .8 - .8] = [0 \quad 0] = 0,
\]

which should come as no surprise.