1. Let $W_1, W_2, \ldots$ be the arrival times in a Poisson process $\{N(t) : t \geq 0\}$ of rate $\lambda > 0$. As usual, $W_{N(t)}$ is the time of the last arrival before time $t$. (Convention: $W_0 = 0$.)

(a) Compute $P[W_{N(t)} \leq x]$ for $0 \leq x \leq t$. [Hint: $\{W_{N(t)} \leq x\} = \{N(x, t] = 0\}$, where $N(x, t] = N(t) - N(x)$ is the number of arrivals in the time interval $(x, t]$.

(b) Use the result of part (a) and the tail-integral formula to find $E[W_{N(t)}]$.

Solution. (a) As hinted, because $N(x, t]$ has the Poisson distribution with mean $\lambda(t - x)$ when $0 \leq x \leq t$,

$$P[W_{N(t)} \leq x] = P[N(x, t] = 0] = e^{-\lambda(t-x)}, \quad 0 \leq x \leq t.$$ Clearly $P[W_{N(t)} \leq x] = 1$ if $x \geq t$.

(b) By the tail-integral formula (and because $P[W_{N(t)} > x] = 0$ if $x \geq t$)

$$E[W_{N(t)}] = \int_0^\infty P[W_{N(t)} > x] \, dx$$

$$= \int_0^t P[W_{N(t)} > x] \, dx$$

$$= \int_0^t (1 - e^{-\lambda(t-x)}) \, dx$$

$$= t - \frac{e^{-\lambda(t-x)}}{\lambda} \bigg|_0^t$$

$$= t - \frac{1 - e^{-\lambda t}}{\lambda}.$$ 

2. Planes arrive at Berghlind Field at the times of a renewal process $N(t)$ with mean interarrival time equal to $\mu$. The number of passengers on plane $k$ is a random variable $C_k$, with mean $\nu = E[C_k]$. Assume that $C_1, C_2, \ldots$ are independent and identically distributed, and independent of the plane-arrival process. As usual, let $M(t) = E[N(t)]$, and let

$$Z(t) = \sum_{k=1}^{N(t)} C_k, \quad t \geq 0,$$ denote the number of passengers that have arrived by time $t$.

(a) Find an expression for $E[Z(t)]$ in terms of $M(t)$ and $\nu$.

(b) Use the renewal theorem to find

$$\lim_{t \to \infty} \frac{E[Z(t)]}{t},$$ the long-term arrival rate of passengers.

Solution. (a) By a formula for the expectation of a random sum of iid random variables (that are independent of the random number of terms) that we saw in Math 180B,

$$E[Z(t)] = E[C_1] \cdot E[N(t)] = \nu \cdot M(t).$$
If you don’t remember this formula, just use the Law of Total Probability:

\[
E[Z(t)] = \sum_{n=0}^{\infty} E \left[ \sum_{k=1}^{n} C_k \mid N(t) = n \right] \cdot P[N(t) = n]
\]

\[
= \sum_{n=0}^{\infty} E \left[ \sum_{k=1}^{n} C_k \right] \cdot P[N(t) = n]
\]

\[
= \sum_{n=0}^{\infty} n\nu \cdot P[N(t) = n]
\]

\[
= \nu \sum_{n=0}^{\infty} n \cdot P[N(t) = n]
\]

\[
= \nu E[N(t)].
\]

(a) By the basic renewal theorem

\[
\lim_{t \to \infty} \frac{M(t)}{t} = \frac{1}{\mu}.
\]

Therefore, using the result of part (a),

\[
\lim_{t \to \infty} \frac{E[Z(t)]}{t} = \frac{\nu}{\mu}.
\]

3. Consider a renewal counting process \( N(t) \) whose interarrival times \( X_1, X_2, \ldots \) have mean \( \mu = E[X_k] \) and finite variance \( \sigma^2 = \text{Var}[X_k] \). Let

\[
\gamma_t = W_{N(t)+1} - t = \min\{s > t : s = W_k \text{ for some } k \geq 1\} - t
\]

be the excess life process. We know from class discussion that the renewal function \( M(t) = E[N(t)] \) satisfies

\[
M(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + \epsilon(t),
\]

where the error term is small: \( \lim_{t \to \infty} \epsilon(t) = 0 \).

(a) Use Wald’s Identity to evaluate \( E[\gamma_t] \).

(b) Find the limit

\[
\lim_{t \to \infty} E[\gamma_t].
\]

Solution. (a) Wald’s Identity tells us that

\[
E[W_{N(t)+1}] = \mu(M(t) + 1)
\]

for all \( t > 0 \). Therefore

\[
E[\gamma_t] = \mu(M(t) + 1) - t, \quad \forall t > 0.
\]
(b) By part (a) and the asymptotic expression given for $M(t)$

$$E[\gamma_t] = \mu \left( \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + \epsilon(t) + 1 \right) - t$$

$$= \frac{\sigma^2 - \mu^2}{2\mu} + \mu \epsilon(t) + \mu$$

$$= \frac{\sigma^2 + \mu^2}{2\mu} + \mu \epsilon(t).$$

Therefore

$$\lim_{t \to \infty} E[\gamma_t] = \frac{\sigma^2 + \mu^2}{2\mu}.$$  

4. Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion. Fix $u > 0$ and define a stochastic process $X$ by the formula

$$X(t) = B(t + u) - B(u), \quad t \geq 0.$$

(a) Compute the covariance function $\text{Cov}[X(s), X(t)]$ for this process.

(b) Explain why $\{X(t) : t \geq 0\}$ is a standard Brownian motion.

Solution. (a) We have, for any $s \geq 0$ and $t \geq 0$,

$$\text{Cov}[X(s), X(t)] = \text{Cov}[B(s + u) - B(u), B(t + u) - B(u)]$$

$$= \text{Cov}[B(s + u), B(t + u)] - \text{Cov}[B(s + u), B(u)] - \text{Cov}[B(u), B(t + u)] + \text{Cov}[B(u), B(u)]$$

$$= \min(s + u, t + u) - u - u + u = \min(s, t) + u - u - u + u$$

$$= \min(s, t).$$

(b) Being a linear combination of values of the Brownian motion $\{B(t) : t \geq 0\}$, the process $\{X(t) : t \geq 0\}$ is a Gaussian process. Also, $Z(0) = B(u) - B(u) = 0$, and $E[X(t)] = E[B(t + u) - B(u)] = 0 - 0 = 0$ for all $t > 0$. And the covariance of $\{X(t) : t \geq 0\}$ is that of a standard Brownian motion, by part (a). Finally, $\{X(t) : t \geq 0\}$ is continuous as a function of $t$. Therefore $\{X(t) : t \geq 0\}$ is a standard Brownian motion.