1. [20 points]. Let \( \{ X(t) : t \geq 0 \} \) be a birth-and-death process with state space \( \{ 0, 1, 2 \} \), and rates \( \lambda_0 = \lambda_1 = 3, \mu_1 = 2, \mu_2 = 4 \). (Also, \( \lambda_2 = \mu_0 = 0 \).) Let \( T_0 = \min \{ t \geq 0 : X(t) = 0 \} \) be the first time the chain is in state 0.

(a) Show that \( P[T_0 < \infty | X(0) = 2] = 1 \). [Hint: Use first-jump analysis to compute \( u_n = P[T_0 < \infty | X(0) = n] \) for \( n = 0, 1, 2 \).]

(b) Find \( E[T_0 | X(0) = 2] \).

Solution. (a) As the jump probabilities are proportional to the jump rates, we have

\[
\begin{align*}
    u_2 &= u_1, \\
    u_1 &= \frac{3}{5} u_2 + \frac{2}{5} u_0, \\
    u_0 &= 1.
\end{align*}
\]

Substituting the first and last equations into the second one, thereby eliminating \( u_2 \) and \( u_0 \) we obtain

\[
    u_1 = \frac{3}{5} u_1 + \frac{2}{5}.
\]

Solving this for \( u_1 \) yields \( u_1 = 1 \). From this and the first equation it follows that \( u_2 = 1 \) as well.

(b) Let \( w_n := E[T_0 | X(0) = n] \). By first jump analysis we have (note that the sojourns in states 2 and 1 are exponential with rates 4 and 5 respectively)

\[
\begin{align*}
    w_2 &= \frac{1}{4} + w_1, \\
    w_1 &= \frac{1}{5} + \frac{3}{5} w_2 + \frac{2}{5} w_0, \\
    w_0 &= 0.
\end{align*}
\]

Because of the third equation, this simplifies to the two-by-two system

\[
\begin{align*}
    w_2 &= \frac{1}{4} + w_1, \\
    w_1 &= \frac{1}{5} + \frac{3}{5} w_2.
\end{align*}
\]

Adding these two equations and canceling \( w_1 \) from both sides we get

\[
    w_2 = \frac{9}{20} + \frac{3}{5} w_2,
\]

and so

\[
    \frac{2}{5} w_2 = \frac{9}{20},
\]

and finally

\[
    w_2 = \frac{9}{8} = 1.125.
\]
2. [20 points]. Let \( \{X(t) : t \geq 0\} \) be a Markov chain with finite state space \( S = \{0, 1, \ldots, N\} \), transition probability matrix \( P(t) = [P_{ij}(t)] \), and infinitesimal matrix \( A = P'(0) \). Let \( \nu = [\nu_0, \nu_1, \ldots, \nu_N] \) be a row vector with \( \sum_{i=0}^{N} \nu_i = 1 \).

(a) Show that if \( \nu P(t) = \nu \) for all \( t \geq 0 \), then \( \nu A = 0 \). [Hint: Differentiate.]

(b) Suppose, conversely, that \( \nu A = 0 \). Show that \( \nu P(t) = \nu \) for all \( t \geq 0 \). [Hint: Consider the vector-valued function \( v(t) = \nu P(t), t \geq 0 \). Show that \( v(t) \) is constant as a function of \( t \) (more differentiation) and then note that \( v(0) = \nu \).]

Solution. (a) Differentiate \( \nu P(t) = \nu \) with respect to \( t \) to see that \( \nu P'(t) = 0 \). Now set \( t = 0 \) and recall that \( P'(0) = A \) to get \( \nu A = 0 \).

(b) Now suppose that \( \nu A = 0 \). Per the Hint, let \( v(t) := \nu P(t) \) for \( t \geq 0 \). Clearly \( v(0) = \nu P(0) = \nu I = \nu \). Also, differentiating as in part (a), and using the backward equation we find that

\[
v'(t) = \nu P'(t) = \nu AP(t) = 0P(t) = 0.
\]

Thus each component of the vector \( v(t) \) is a constant function of time, so \( v(t) \) is constant as a function of time, so \( v(t) = v(0) = \nu \) for all \( t \geq 0 \).

3. [20 points]. Consider a continuous time Markov chain \( \{X(t) : t \geq 0\} \) with state space \( \{0, 1\} \), transition probability matrix \( P(t) = [P_{ij}(t)] \), and infinitesimal matrix

\[
A = P'(0) = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix},
\]

where \( 0 < \alpha, \beta < 1 \) and \( \alpha \neq \beta \). Assuming that \( X(0) = 0 \), the successive times at which \( X \) returns to the state 0 form a renewal process with inter-arrival time density \( f \). [A formula for \( f \) appears below in the Hint for part (b).]

(a) Use first-jump analysis to argue that the transition probability \( P_{00}(t) = P[X(t) = 0|X(0) = 0] \) satisfies

\[
P_{00}(t) = h(t) + \int_{0}^{t} P_{00}(s)f(t-s) \, ds, \quad t > 0,
\]

where \( h(t) = e^{-\alpha t} \).

(b) Use the Key Renewal Theorem and the result of part (a) to deduce that

\[
\lim_{t \to \infty} P_{00}(t) = \beta/(\alpha + \beta).
\]

[Hint: Note that \( \int_{0}^{\infty} tf(t) \, dt = \alpha^{-1} + \beta^{-1} \). (Why?) The following formula

\[
f(t) = \int_{0}^{t} a e^{-\alpha s} \beta e^{-\beta (t-s)} \, ds = \alpha \beta \left( \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha} \right), \quad t \geq 0.
\]

for the interarrival-time density \( f \) may be useful.]

Solution. (a) Let \( S_0 \) be the initial sojourn time in state 0 and let \( R = \min\{t > S_0 : X(t) = 0\} \) be the return time to state 0. (This is the first arrival time in the renewal process mentioned in
the statement of the problem.) To be in state 0 at time \( t \), either \( S_0 \geq t \) (which has probability \( h(t) = e^{-\alpha t} \)) or \( S_0 < t \), and then in order for \( X(t) \) to be back in state 0, we must even have \( R \leq t \). Conditioning on the value of \( R \) (noting that \( R \) has density \( f \)), the probability of this latter event is therefore

\[
\int_0^t f(s)P[X(t) = 0|R = s] \, ds.
\]

By the Markov property, the conditional probability \( P[X(t) = 0|R = s] \) is equal to \( P[X(s) = 0] = P_{00}(t-s) \). Combining these observations we obtain

\[
P_{00}(t) = h(t) + \int_0^t P_{00}(t-s)f(s) \, ds = \int_0^t P_{00}(s)f(t-s) \, ds, \quad t > 0.
\]

(b) Clearly \( R \) is the sum \( S_0 + S_1 \) of sojourn times that are exponential with respective rates \( \alpha \) and \( \beta \). Therefore \( \mu = E[R] = \alpha^{-1} + \beta^{-1} = (\alpha + \beta)/\alpha\beta \). By the Key Renewal Theorem, applied with \( H(t) = P_{00}(t) \) and \( h(t) = e^{-\alpha} \),

\[
\lim_{t \to \infty} P_{00}(t) = \frac{\int_0^\infty h(t) \, dt}{\mu} = \frac{\alpha^{-1}}{(\alpha + \beta)/\alpha\beta} = \frac{\beta}{\alpha + \beta}.
\]

4. [20 points]. Consider a renewal process with i.i.d. inter-arrival times \( X_1, X_2, \ldots \), each having mean \( \mu \) and (finite) variance \( \sigma^2 > 0 \). Suppose that the renewal function is given by the formula

\[
M(t) = \frac{2t}{3} - \frac{2}{9}(1 - e^{-3t}), \quad t \geq 0.
\]

(a) Find the value of \( \mu \).

(b) Find the value of \( \sigma^2 \).

Solution. (a) Because \( \mu^{-1} = \lim_{t \to \infty} \frac{M(t)}{t} = \frac{2}{3} \) (by the formula for \( M(t) \)) we must have \( \mu = 3/2 \).

(b) We know that when \( X_1 \) has finite variance,

\[
\lim_{t \to \infty} \left[ M(t) - \frac{t}{\mu} \right] = \frac{\sigma^2 - \mu^2}{2\mu^2}.
\]

By part (a) and the formula for \( M(t) \), \( \lim_{t \to \infty} [M(t) - t/\mu] = -2/9 \). It follows that

\[
-\frac{2}{9} = \frac{\sigma^2 - 9/4}{9/2}.
\]

Solving this for \( \sigma^2 \) we get

\[
\sigma^2 = \frac{5}{4}.
\]

5. [20 points]. Let \( \{B(t) : t \geq 0\} \) be a standard Brownian motion (with \( B(0) = 0 \)). For \( b \neq 0 \), let \( T_b = \min\{t > 0 : B(t) = b + t\} \) be the first time the Brownian motion hits the line with slope 1 and intercept \( b \). Find the probability that \( B(t) \) hits the upper line first; that is, find

\[
P[T_1 < T_{-1}].
\]
[Hint: Use the Brownian-motion-with-drift \( X(t) = B(t) - t \).]

**Solution.** This is just the Gambler’s Ruin formula for the generalized Brownian motion with \( x = 0 \), \( \mu = -1 \), \( \sigma^2 = 1 \), with barriers at \(-1\) and \(+1\). For this choice of parameter values we have

\[
S(x) = \exp(-2\mu x/\sigma^2) = e^{2x}.
\]

Therefore

\[
P[T_1 < T_{-1}] = \frac{S(0) - S(-1)}{S(1) - S(-1)} = \frac{1 - e^{-2}}{e^2 - e^{-2}} = 0.1192....
\]

6. **[20 points]**. If \( \{B(t) : t \geq 0\} \) is a standard Brownian motion, then

\[
Y(t) = \begin{cases} (1 - t)B(t/(1 - t)), & 0 \leq t < 1, \\ 0, & t = 1, \end{cases}
\]

defines a Brownian Bridge process. (You may take this fact for granted.) Using this representation, show that if \( 0 < a < b < 1 \) then

\[
P[Y(t) = 0 \text{ for some } t \in (a, b)] = \frac{2}{\pi} \arcsin \sqrt{\frac{b - a}{b(1 - a)}}.
\]

**Solution.** Notice that \( Y(t) = 0 \) if and only if \( B(t/(1 - t)) = 0 \), and that \( a < t \leq b \) if and only if \( a/(1 - a) < t/(1 - t) \leq b/(1 - b) \) (the latter because \( t \mapsto t/(1 - t) \) is monotone increasing). Relabelling \( t/(1 - t) \) as \( s \) we see that

\[
P[Y(t) = 0 \text{ for some } t \in (a, b)] = P[B(s) = 0 \text{ for some } s \in (a/(1 - a), b/(1 - b))]
\]

\[
= \frac{2}{\pi} \arctan \sqrt{\frac{b/(1 - b) - a/(1 - a)}{a/(1 - a)}}
\]

\[
= \frac{2}{\pi} \arctan \sqrt{\frac{(1 - a)b - a(1 - b)}{a(1 - b)}}
\]

\[
= \frac{2}{\pi} \arctan \sqrt{\frac{b - a}{a(1 - b)}}
\]

\[
= \frac{2}{\pi} \arcsin \sqrt{\frac{b - a}{b(1 - a)}},
\]

the final equality resulting from the Pythagorean Theorem (and a diagram).

7. **[20 points]**. Let \( \{B(t) : t \geq 0\} \) be a standard Brownian motion (with \( B(0) = 0 \)).

(a) Find the distribution of the average \( (B(2) + B(3))/2 \) of the positions of the Brownian motion at times 2 and 3. That is, express the probability

\[
P \left[ \frac{B(2) + B(3)}{2} \leq x \right]
\]

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in terms of the standard normal cdf $\Phi$.

(b) Define $M(t) = \max\{B(u) : 0 \leq u \leq t\}$. Which of the two probabilities

$$P[M(3) \leq 1], \quad P[B(3) \leq 1]$$

is larger? Explain. (You do not need the table of the normal cdf to answer this question.)

Solution. (a) The mean of $(B(2) + B(3))/2$ is zero and the variance is $[2 + 3 + 2 \cdot 2]/4 = 9/4 = 2.25$. Therefore the average in question is normally distributed with mean 0 and variance 2.25 (hence standard deviation 1.5):

$$P \left[ \frac{B(2) + B(3)}{2} \leq x \right] = P \left[ \frac{B(2) + B(3)}{2 \cdot 1.5} \leq \frac{x}{1.5} \right] = \Phi(x/1.5).$$

(b) We know that $M(3)$ has the same distribution as $|B(3)|$. Therefore

$$P[M(3) \leq 1] = P[|B(3)| \leq 1] = P[-1 \leq B(3) \leq 1] < P[B(3) \leq 1],$$

so the second of the two mentioned probabilities is the larger.

8. [30 points]. This problem concerns an $M/M/1$ queue with “balking.” Customers arrive at a service counter in accordance with a Poisson process of rate $\lambda$; there is a single server and the service times are exponentially distributed with rate $\mu$. If an arriving customer finds the server busy, the customer either joins the queue (with probability $p$) or leaves immediately (in a huff, with probability $q = 1 - p$). (The customer always stays if the server is idle.) Let $X(t)$ be the queue size at time $t$. The process $\{X(t) : t \geq 0\}$ is a birth-and-death process.

(a) Give explicit formulas (in terms of $\lambda$, $\mu$, and $p$) for the birth rates $\lambda_n, n = 0,1,\ldots$ and the death rates $\mu_n, n = 1,2,\ldots$.

(b) Under what conditions on $\lambda$, $\mu$, and $p$ will there be a stationary distribution for $X(t)$.

(c) Find a formula for the stationary distribution $\pi_j, j = 0,1,2,\ldots$, when it exists.

Solution. (a) $\lambda_0 = \lambda$ but $\lambda_n = \lambda p$ if $n \geq 1$. And $\mu_n = \mu$ for all $n \geq 1$.

(b) In view of part (a) we have $\theta_n = p^{-1}(\lambda p/\mu)^n$ for $n \geq 1$. (As usual, $\theta_0 = 1$.) The series $\sum_n \theta_n$ is a geometric series that converges if and only if the ratio between successive terms is less than 1; that is, if and only if $\lambda p/\mu < 1$, or (what is the same) $\lambda p < \mu$.

(c) Adding the geometric series we get

$$\sum_{n=0}^{\infty} \theta_n = 1 + \sum_{n=1}^{\infty} p^{-1}(\lambda p/\mu)^n = 1 + p^{-1} \frac{\lambda p/\mu}{1 - \lambda p/\mu} = 1 + \frac{\lambda}{\mu - \lambda p} = \frac{\mu + \lambda q}{\mu - \lambda p}.$$ 

Therefore

$$\pi_n = \begin{cases} \frac{\mu - \lambda p}{\mu + \lambda q}, & n = 0, \\ \frac{\mu - \lambda p}{(\mu + \lambda q)p} \left( \frac{\lambda p}{\mu} \right)^n, & n \geq 1. \end{cases}$$