1. [20 points]. Consider a two-phase renewal process, in which the inter-arrival times can be expressed as \( X_k = Y_k + Z_k \), with i.i.d. pairs \((Y_1, Z_1), (Y_2, Z_2), \ldots\). Let \( \alpha = \text{E}[Y_k], \beta = \text{E}[Z_k] \), and \( \mu = \alpha + \beta \). There is a cost \( A \) per unit of time associated with the “Y phase” and a cost \( B \) per unit of time associate with the “Z phase”. Find the long-run cost per unit of time of this system.

Solution. The long-run fraction of time spent in the “Y-phase” is \( \frac{\alpha}{\mu} \) and the long-run fraction of time spent in the “Z-phase” is \( \frac{\beta}{\mu} \). The long-run cost per unit time is therefore the weighted average of the separate costs \( A \) and \( B \); that is: \( \frac{\alpha A + \beta B}{\mu} \).

2. [30 points]. Consider a renewal process for which the inter-arrival times \( X_1, X_2, \ldots \), have density function \( f \). Assume that each \( X_k \) has finite mean \( \mu \) and finite variance \( \sigma^2 \). Recall that \( \beta_t = W_{N(t)+1} - W_{N(t)} \) is the total length of the renewal cycle in progress at time \( t \). Let \( H(t) = \text{E}[\beta_t] \) for \( t > 0 \).

(a) Explain why \( H \) satisfies the renewal equation

\[
H(t) = h(t) + \int_0^t H(t-s) f(s) \, ds, \quad t > 0,
\]

where \( h(t) = \text{E}[X_1; X_1 > t] = \int_t^{\infty} x f(x) \, dx \).

(b) Verify that \( \int_0^{\infty} h(t) \, dt = \sigma^2 + \mu^2 \).

(c) Use the results of parts (a) and (b), and the Renewal Theorem, to find

\[
\lim_{t \to \infty} \text{E}[\beta_t].
\]

Solution. (a) If \( X_1 > t \) then \( \beta_t = X_1 \). If \( X_1 = s \leq t \) then the conditional expectation of \( \beta_t \), given that value of \( X_1 \), is equal to the mean of \( \beta'_{t-s} \), where \( \beta'_{t-s} \) is the total length of the renewal cycle in progress at time \( t - s \), for the (shifted) renewal process \( W'_n := W_{n+1} - W_1 : n \geq 1 \). Thus

\[
H(t) = \text{E}[\beta_t]
= \text{E}[X_1; X_1 > t] + \int_0^t \text{E}[\beta_t \mid X_1 = s] f(s) \, ds
= \text{E}[X_1; X_1 > t] + \int_0^t \text{E}[\beta'_{t-s} \mid f(s)] \, ds
= h(t) + \int_0^t H(t-s) f(s) \, ds.
\]

(b) Because \( \text{E}[X_1; X_1 > t] = \int_t^{\infty} x f(x) \, dx \), we have:

\[
\int_0^{\infty} h(t) \, dt = \int_0^{\infty} \int_t^{\infty} x f(x) \, dx \, dt
= \int_0^{\infty} \int_0^{x} x f(x) \, dt \, dx
= \int_0^{\infty} x^2 f(x) \, dx
= \text{E}[X^2] = \sigma^2 + \mu^2.
\]
(c) In the present situation, the function $h$ is non-negative, bounded above by $E[X_1]$, and decreasing. Therefore, by (a) and (b) and the Key Renewal Theorem

$$
\lim_{t \to \infty} E[\beta_t] = \int_0^\infty \frac{h(t)}{\mu} \, dt = \frac{\sigma^2 + \mu^2}{\mu} = \mu + \frac{\sigma^2}{\mu}.
$$

3. [30 points]. Let $N(t), t \geq 0,$ be a renewal counting process whose inter-arrival times $X_1, X_2, \ldots$ have the exponential density with parameter $\lambda > 0$. As usual, $W_k = X_1 + \cdots + X_k$ for $k = 1, 2, \ldots$. Suppose $0 < s \leq t$.

(a) Show that $\{X_1 \leq s, N(t) = 1\} = \{N(s) = 1, N(s, t] = 0\}$, where $N(s, t] = \# \{k \geq 1 : s < W_k \leq t\}$ is the number of renewals in the time interval $(s, t]$.

(b) Explain why $N(s)$ and $N(s, t]$ are independent random variables, with Poisson distributions of parameter $\lambda s$ and $\lambda (t - s)$ respectively.

(c) Deduce from (a) and (b) that $P[X_1 \leq s | N(t) = 1] = \frac{s}{t}$, $0 < s \leq t$;

that is, the conditional distribution of $X_1$, given that there is precisely one renewal before time $t$, is uniform on $(0, t]$.

Solution. (a) To say that $X_1 \leq s$ and that $N(t) = 1$ is to say that the first renewal occurs at or before time $s$ but that no further renewals occur before time $t$. On the other hand, to say that $N(s) = 1$ and $N(s, t] = 0$ is to say that there is exactly one renewal in $(0, s]$ (so that $X_1 \leq s$) but no renewals in $(s, t]$ (so no further renewals before time $t$). This shows that $\{X_1 \leq s, N(t) = 1\} = \{N(s) = 1, N(s, t] = 0\}$.

(b) Because the inter-arrival times in our renewal process are exponential with parameter $\lambda$, the renewal process under consideration is in fact a Poisson process with rate $\lambda$. This means that $N(s)$ has the Poisson distribution with parameter $\lambda s$, $N(s, t]$ has the Poisson distribution with parameter $\lambda (t - s)$, and that $N(s)$ and $N(s, t]$ are independent (as they count arrivals in disjoint time intervals).

(c) By (b): $P[N(s) = 1] = \lambda se^{-\lambda s}$, $P[N(s, t] = 0] = e^{-\lambda (t-s)}$, and $P[N(t) = 1] = \lambda te^{-\lambda t}$.

Because of the independence asserted in (b),

$$
P[X_1 \leq s | N(t) = 1] = \frac{P[N(s) = 1, N(s, t] = 0]}{P[N(t) = 1]} = \frac{\lambda se^{-\lambda s} \cdot e^{-\lambda (t-s)}}{\lambda te^{-\lambda t}} = \frac{s}{t},
$$

for $0 < s \leq t$.

4. [20 points]. Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion. Define a stochastic process $\{X(t) : t \geq 0\}$ by the formulas

$$
X(t) = tB(1 + t^{-1}) - tB(1), \quad t > 0,
$$

$X(0) = 0$, 

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You may take for granted the fact that $\lim_{t \to 0^+} X(t) = 0$, with probability 1.

(a) Compute the covariance function $\text{Cov}[X(s), X(t)]$ for this process.

(b) Explain why $\{X(t) : t \geq 0\}$ is a standard Brownian motion.

Solution. (a) We have, for $0 < s \leq t$,

$$\text{Cov}[X(s), X(t)] = \text{Cov}[s(B(1 + s^{-1}) - B(1), t(B(1 + t^{-1}) - B(1))]$$

$$= st \text{Cov}[B(1 + s^{-1}) - B(1), B(1 + t^{-1}) - B(1)]$$

$$= st(\text{Cov}[B(1 + s^{-1}, B(1 + t^{-1})] - \text{Cov}[B(1 + s^{-1}, B(1)]$$

$$- \text{Cov}[B(1), B(1 + t^{-1})] + \text{Cov}[B(1), B(1)])$$

$$= st((1 + t^{-1}) - 1 - 1 + 1) = st(t^{-1}) = s = \min(s, t).$$

(b) The stochastic process $X = \{X(t) : t \geq 0\}$ is formed by taking linear combinations of random variables from the Brownian motion $\{B(t) : t \geq 0\}$. Therefore $X$ is a Gaussian process. Clearly $E[X(t)] = 0$ for each $t$, so (in view of part (a)), $X$ has the same mean and covariance functions as a Brownian motion. Finally, $X(0) = 0$ and $t \mapsto X(t)$ is continuous (at least with probability 1). The latter assertion is clear for $t > 0$ by composition of continuous functions; for $t = 0$ the continuity follows from the stipulation made in the statement of the problem.