1. Consider a birth-and-death process $X(t)$ with state space $\{0, 1, 2, 3\}$, birth rates $\lambda_k = 1$ for $k = 0, 1, 2$, and death rates $\mu_k = k$ for $k = 1, 2, 3$ ($\mu_0 = \lambda_3 = 0$). Let $T_0 = \min\{t > 0 : X(t) = 0\}$. You may assume that $P[T_0 < \infty | X(0) = 3] = 1$.

   (a) Calculate $E[T_0 | X(0) = 3]$. [Hint: Use first-jump analysis to find a system of linear equations for $v_i := E[T_0 | X(0) = i]$ for $i = 0, 1, 2, 3$. Clearly $v_0 = 0$.]

   (b) Find the limit distribution $\pi_j = \lim_{t \to \infty} P_{ij}(t)$, $j = 0, 1, 2, 3$, for this process.

Solution. (a) By first-jump analysis we have (keeping in mind that $v_0 = 0$)

\[
\begin{align*}
  v_1 &= \frac{1}{2} + \frac{1}{2} v_2 \\
  v_2 &= \frac{1}{3} + \frac{2}{3} v_1 + \frac{1}{3} v_3 \\
  v_3 &= \frac{1}{3} + v_2
\end{align*}
\]

(For example, starting in state 2 we hold there for an exponential time with rate $1 + 2 = 3$, and then jump: up to 3 with probability $1/(1 + 2) = 1/3$ and down to 1 with probability $2/(1 + 2) = 2/3$.) Substituting the third equation into the second (thereby eliminating $v_3$) and then clearing out denominators we obtain

\[
\begin{align*}
  2v_1 &= 1 + v_2 \\
  6v_2 &= 4 + 6v_1
\end{align*}
\]

The solution of this two-by-two solution is easily seen to be

\[
\begin{align*}
  v_1 &= \frac{5}{3} \\
  v_2 &= \frac{7}{3}
\end{align*}
\]

Finally,

\[
E[T_0 | X(0) = 3] = v_3 = \frac{1}{3} + v_2 = \frac{8}{3}.
\]

(b) For this finite state space Birth-and-Death process we have

\[
\begin{align*}
  \theta_0 &= 1 \\
  \theta_1 &= \frac{1}{1} = 1 \\
  \theta_2 &= \frac{1 \cdot 1}{1 \cdot 2} = \frac{1}{2} \\
  \theta_3 &= \frac{1 \cdot 1 \cdot 1}{1 \cdot 2 \cdot 3} = \frac{1}{6}.
\end{align*}
\]

The sum of the $\theta_j$s is $8/3$, and using this to normalize we obtain the equilibrium distribution $\pi$:

\[
\pi = [3/8 \ 3/8 \ 3/16 \ 1/16].
\]
From the general theorem about limit probabilities for Birth-and-Death chains, we know that
\[ \lim_{t \to \infty} P_{ij}(t) = \pi_j, \]
for all \( i \) and \( j \), where \( \pi_j \) is as computed above.

2. Consider a pure birth process \( X(t) \) with state space \( S = \{0, 1, 2, 3, 4\} \), and with \( P[X(0) = 0] = 1 \). Suppose the birth rates are \( \lambda_0 = 4, \lambda_1 = 2, \lambda_2 = 6, \lambda_3 = 1, \lambda_4 = 0 \). (Note that 4 is an absorbing state.) Let \( T_4 := \min\{t : X(t) = 4\} \) be the first time the population size reaches 4.
   (a) Express \( T_4 \) in terms of (some or all of) the sojourn times \( S_0, S_1, S_2, S_3, S_4 \).
   (b) Find \( E[T_4] \).
   (c) Find \( \text{Var}[T_4] \).

Solution. (a) \( T_4 = S_0 + S_1 + S_2 + S_3 \).
(b) Because the mean of an exponential random variable is the reciprocal of its rate, we have
\[ E[T_4] = \sum_{k=0}^{3} E[S_k] = \frac{1}{4} + \frac{1}{2} + \frac{1}{6} + 1 = \frac{23}{12} = 1.916666... \]
(c) Because the variance of an exponential random variable is the reciprocal of the square of its rate, and because the sojourn times are independent, we have
\[ \text{Var}[T_4] = \sum_{k=0}^{3} \text{Var}[S_k] = \frac{1}{16} + \frac{1}{4} + \frac{1}{36} + 1 = \frac{193}{144} = 1.3402777.... \]

3. A Markov chain \( X(t), t \geq 0 \), has state space \( \{0, 1\} \) and transition probability matrix
\[ P(t) = \begin{bmatrix} .6 + .4e^{-5t} & .4 - .4e^{-5t} \\ .6 - .6e^{-5t} & .4 + .6e^{-5t} \end{bmatrix}. \]
(a) Compute the infinitesimal matrix \( A = P'(0) \).
(b) Let \( \pi = (\pi_0, \pi_1) \) be the top row of the limit matrix \( \lim_{t \to \infty} P(t) \). Compute \( \pi A \).
(c) Using the given formula for \( P(t) \), show that \( P'(t) = AP(t) \) for all \( t > 0 \).

Solution. (a) Differentiating:
\[ P'(t) = \begin{bmatrix} -2e^{-5t} & 2e^{-5t} \\ 3e^{-5t} & -3e^{-5t} \end{bmatrix}. \]
In particular,
\[ A = P'(0) = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}. \]
(b) Because \( \lim_{t \to \infty} e^{-5t} = 0 \), the top row of \( P(t) \) converges to \( \pi = [.6, .4] \). The matrix product is
\[ \pi A = [.6, .4] \cdot \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} = [.6(-2) + .4(3), .6(2) + .4(-3)] = [.0, 0]. \]
as expected.

(c) Multiply:

\[
\mathbf{AP}(t) = \begin{bmatrix} 2 & 2 \\ 3 & -3 \end{bmatrix} \cdot \begin{bmatrix} .6 + .4e^{-5t} & .4 - .4e^{-5t} \\ .6 - .6e^{-5t} & .4 + .6e^{-5t} \end{bmatrix}
\]

\[
= \begin{bmatrix} -1.2 - .8e^{-5t} + 1.2 - 1.2e^{-5t} & -.8 + .8e^{-5t} + .8 + 1.2e^{-5t} \\ 1.8 + 1.2e^{-5t} - 1.8 + 1.8e^{-5t} & 1.2 - 1.2e^{-5t} - 1.2 - 1.8e^{-5t} \end{bmatrix}
\]

\[
= \begin{bmatrix} -2e^{-5t} & 2e^{-5t} \\ 3e^{-5t} & -3e^{-5t} \end{bmatrix},
\]

which coincides with \( \mathbf{P}'(t) \) as computed earlier for part (a).

4. A factory has an unlimited number of machines and a single repairman. At any given time all but finitely many of the machines are “broken” and awaiting repair, the rest are operating. The time-until-failure of an operating machine is an exponentially distributed random variable with parameter 2. The repair times are exponentially distributed with parameter 3. The failures of the different machines are independent, and at most one machine is under repair at any time. We model the number of operating machines as a birth-and-death process \( X(t) \) with state space \( \{0, 1, 2, \ldots \} \).

(a) Find the birth rates \( \lambda_n \) and the death rates \( \mu_n \) for \( X(t) \).

(b) Find the stationary distribution for this birth-and-death process.

(c) In the long run, for what fraction of the time are all of the machines broken?

Solution. (a) Evidently \( \lambda_n = 3 \) because there is a single rate-3 repairman, and a machine being fixed corresponds to a “birth”. And \( \mu_n = 2n \) because when the state of the system is \( n \), there are \( n \) machines in operation, each having a failure rate of 2, and a machine breaking corresponds to a “death”.

(b) We have \( \theta_j = \frac{3^j}{2^j} \) for \( j = 0, 1, 2, \ldots \). The sum of these is an exponential series with value \( e^{3/2} \). Therefore \( \pi_j = e^{-3/2} \left( \frac{3^j}{2^j} \right) \), \( j = 0, 1, 2, \ldots \) — the Poisson distribution with parameter \( 3/2 \).

(c) This is just \( \pi_0 = e^{-3/2} = 0.2231 \ldots \).