1. [30 points]. Let \( \{N(t) : t \geq 0\} \) be a renewal process with arrival times \( W_1, W_2, \ldots \). Suppose that the renewal density \( m \) has the form \( m(t) = 1 - e^{-4t} \) for \( t \geq 0 \).

(a) Find \( \mu \), the mean inter-arrival time.
(b) Find \( E[W_{N(t)+1}] \).
(c) Find \( \lim_{t \to \infty} E[\gamma_t] \), where \( \gamma_t \) is the excess life process associated with the renewal process.

Solution. (a) Integrating we find that
\[
M(t) = \int_0^t m(s) \, ds = \int_0^t (1 - e^{-4s}) \, ds = t - \frac{1}{4} (1 - e^{-4t}).
\]
Consequently,
\[
\frac{1}{\mu} = \lim_{t \to \infty} M(t) = 1,
\]
and so \( \mu = 1 \).

Alternatively, we know that \( \lim_{t \to \infty} m(t) = 1/\mu \), and since the limit is clearly 1, we have \( \mu = 1 \) as before.

(b) By Wald’s Identity,
\[
E[W_{N(t)+1}] = \mu \cdot (M(t) + 1) = t - \frac{1}{4} (1 - e^{-4t}) + 1 = t + \frac{3}{4} + \frac{1}{4} e^{-4t}.
\]

(c) Using (a) and (b), because \( \gamma_t = W_{N(t)+1} - t \), we have
\[
E[\gamma_t] = t + \frac{3}{4} + \frac{1}{4} e^{-4t} - t = \frac{3}{4} + \frac{1}{4} e^{-4t},
\]
and so
\[
\lim_{t \to \infty} E[\gamma_t] = \frac{3}{4}.
\]

2. [30 points]. Let \( \{N(t) : t \geq 0\} \) be a renewal counting process whose inter-arrival times \( X_1, X_2, \ldots \) have the exponential density with parameter \( \lambda > 0 \). As usual, \( W_k = X_1 + \cdots + X_k \) for \( k = 1, 2, \ldots \). Suppose \( 0 < s \leq t \).

(a) Explain why \( \{X_1 \leq s, N(t) = 1\} = \{N(s) = 1, N(s, t] = 0\} \), where \( N(s, t] = \#\{k \geq 1 : s < W_k \leq t\} \) is the number of renewals in the time interval \( (s, t] \).
(b) Explain why \( N(s) \) and \( N(s, t] \) are independent random variables, with Poisson distributions of parameter \( \lambda s \) and \( \lambda (t-s) \) respectively.
(c) Deduce from (a) and (b) that
\[
P[X_1 \leq s|N(t) = 1] = \frac{s}{t}, \quad 0 < s \leq t;
\]
that is, the conditional distribution of \( X_1 \), given that there is precisely one renewal before time \( t \), is uniform on \( (0, t] \).

Solution. (a) \( X_1 \leq s \) if and only if \( N(s) \geq 1 \). In conjunction with \( N(t) = 1 \), we must then have \( N(s) = 1 \) but then no more arrivals before time \( t \), so that \( N(s, t] = 0 \). That is,
\[
\{X_1 \leq s, N(t) = 1\} = \{N(s) = 1, N(s, t] = 0\}.
\]

(b) Because the inter-arrival times are exponential with parameter \( \lambda \), this renewal process is a Poisson process with rate \( \lambda \). This tells us that (i) \( N(s) \) has the Poisson distribution with parameter \( \lambda s \), (ii) \( N(s, t] \)
has the Poisson distribution with parameter $\lambda(t - s)$, and (iii) the random variable $N(s)$ and $N(s,t]$ are independent.

(c)) We now use (a) and (b) to compute

$$P[X_1 \leq s \mid N(t) = 1] = \frac{P[X_1 \leq s, N(t) = 1]}{P[N(t) = 1]} = \frac{P[N(s) = 1, N(s,t] = 0]}{P[N(t) = 1]} = \frac{P[N(s) = 1] \cdot P[N(s,t] = 0]}{P[N(t) = 1]} = \frac{\lambda se^{-\lambda s} \cdot e^{-\lambda(t-s)}}{\lambda e^{-\lambda t}} = \frac{se^{-\lambda t}}{t}$$

as requested. This (conditional) cdf is that of a random variable uniformly distributed over $(0,1)$.

3. [30 points]. Let $\{N(t) : t \geq 0\}$, be a renewal counting process in which the inter-arrival times are uniformly distributed on $(0,1)$. Let $f$ be the inter-arrival time density function. At each renewal time $W_k$ a non-random impulse is created. The impulse dissipates with time: At time $t > W_k$ the magnitude of the impulse created at time $W_k$ has fallen to $\exp(-A(t-W_k))$, where $A > 0$ is a constant. At time $t$, the total magnitude of all impulses created prior to $t$ is

$$Z(t) = \sum_{k:W_k \leq t} \exp(-A(t-W_k)),$$

where the summation extends over all indices $k$ such that $W_k \leq t$.

(a) Define $h(t) := \int_0^t f(s) \cdot e^{-A(t-s)} ds$, and let $H(t) = E[Z(t)]$. By conditioning on the time of the first arrival, show that

$$H(t) = h(t) + \int_0^t f(s) H(t-s) ds.$$

(b) Use the result of part (a) and the Key Renewal Theorem to conclude that $\lim_{t \to \infty} E[Z(t)] = 2/A$.

Solution. (a) The renewal equation is gotten by conditioning on the time of the first arrival. Notice that if $W_1 > t$ then $Z(t) = 0$. Therefore

$$H(t) = E[Z(t)] = \int_0^t E[Z(t) \mid W_1 = s] f(s) ds$$

$$= \int_0^t \left( e^{-A(t-s)} + E[Z(t-s)] \right) f(s) ds.$$  

Here $\tilde{Z}(u) = \sum_{k:W_k \leq u} e^{-A(u-W_k)}$ and $\tilde{W}_k = W_{k+1} - W_1$ is the shifted renewel process, viewed from time $W_1$ forward. As discussed in class, $\tilde{W}_1, \tilde{W}_2, \ldots$ is a renewal process (independent of $W_1$) with the same distribution as the original renewal process. Therefore $E[Z(t-s)] = E[Z(t-s)] = H(t-s)$. Continuing the preceding computation:

$$H(t) = \int_0^t \left( e^{-A(t-s)} + H(t-s) \right) f(s) ds$$

$$= h(t) + \int_0^t H(t-s) f(s) ds,$$
as claimed.

(b) Because the $X_k$ are uniform on $(0, 1)$, $\mu = E[X_k] = 1/2$. For the same reason ($f$ is the uniform on $(0, 1)$ density) a straightforward integration leads to

$$h(t) = \begin{cases} 
(1 - e^{-At})/A, & 0 \leq t \leq 1; \\
e^{-At}(e^A - 1)/A, & t \geq 1.
\end{cases}$$

In particular, $h$ in non-negative and eventually decreasing, so the Key Renewal Theorem can be applied. To that end we need to integrate $h$:

$$\int_0^\infty h(t) \, dt = \int_0^1 \frac{1 - e^{-At}}{A} \, dt + \int_1^\infty e^{-At} \frac{e^A - 1}{A} \, dt$$

$$= A^{-1} \left\{ \int_0^1 (1 - e^{-At}) \, dt + (e^A - 1) \int_1^\infty e^{-At} \, dt \right\}$$

$$= A^{-1} \left\{ 1 - (1 - e^{-A})/A + (e^A - 1)e^{-A}/A \right\}$$

$$= A^{-1} \left\{ 1 - 1/A + e^{-A}/A + 1/A - e^{-A}/A \right\}$$

$$= A^{-1}.$$

Therefore

$$\lim_{t \to \infty} E[Z(t)] = \lim_{t \to \infty} H(t) = \frac{\int_0^\infty h(t) \, dt}{\mu} = \frac{1/A}{1/2} = \frac{2}{A}.$$

4. [30 points]. Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion. Fix $u > 0$ and define a new stochastic process $X$ by the formula

$$X(t) = B(t + u) - B(u), \quad t \geq 0.$$

(a) Compute $E[X(t)]$ for $t > 0$.

(b) Compute the covariance function Cov[$X(s), X(t)$] for this process.

(c) Explain why $\{X(t) : t \geq 0\}$ is a standard Brownian motion.

Solution. (a) $E[X(t)] = E[B(t + u) - B(u)] = E[B(t + u)] - E[B(u)] = 0 - 0 = 0$.

(b) Remember that Cov[$B(u), B(v)$] = $u \wedge v$. Therefore

$$\text{Cov}[X(s), X(t)] = \text{Cov}[B(s + u) - B(u), B(t + u) - B(u)]$$

$$= \text{Cov}[B(s + u), B(t + u)] - \text{Cov}[B(s + u), B(u)]$$

$$- \text{Cov}[B(u), B(t + u)] + \text{Cov}[B(u), B(u)]$$

$$= (s + u) \wedge (t + u) - u - u + u$$

$$= (s + u) \wedge (t + u) - u$$

$$= s \wedge t.$$

(c) The process $X$ is a (non-random) linear transformation of a Gaussian process (the original Brownian motion $B$) and so is itself a Gaussian processes. Clearly $X(t)$ is continuous as a function of $t$. And $X$ has the same mean and covariance functions as the Brownian motion does. Therefore $X$ must be a Brownian motion.