Math 180C, Spring 2019

Supplement on the Renewal Equation

0. These remarks supplement our text and set down some of the material discussed in my lectures. Unexplained notation is as in the text or in lecture.

1. In various contexts (continuous time Markov chains, renewal processes, etc.) we encounter probabilities or expectations that depend on time t and satisfy a *renewal equation*:

(1.1) $$H(t) = h(t) + \int_0^t H(t-s)f(s)\,ds, \qquad t \geq 0,$$

or, briefly,

(1.2) $$H = h + H * f.$$

Here f is a probability density function on $[0,\infty)$, $h : [0,\infty) \to \mathbf{R}$ is a bounded function (typically continuous), and the function H is the "unknown".

2. Example. The renewal function $M(t) := \mathbf{E}[N(t)]$ satisfies

$$M(t) = \sum_{n=1}^{\infty} \mathbf{P}[W_n \leq t] = \sum_{n=1}^{\infty} F_n(t),$$

where the cdf $F_n(t) = \mathbf{P}[W_n \leq t]$ satisfies (condition on the value of X_1, using the Law of Total Probability)

$$F_n(t) = \int_0^t F_{n-1}(t-s)f(s)\,ds = F_{n-1} * f(t).$$

Therefore, because $F_1 = F$,

$$\begin{aligned} M(t) &= F(t) + \sum_{n=2}^{\infty} \int_0^t F_{n-1}(t-s)f(s)\,ds \\ &= F(t) + \sum_{k=1}^{\infty} \int_0^t F_k(t-s)f(s)\,ds \\ &= F(t) + \int_0^t M(t-s)f(s)\,ds; \end{aligned}$$

that is,

(2.1) $$M = F + M * f,$$

a special case of (1.2) when $h = F$. Integrating by parts we can see that

$$\begin{aligned} M * f(t) &= \int_0^t M(t-s)f(s)\,ds \\ &= M(t-s)F(s)\Big|_{s=0}^{s=t} + \int_0^t m(t-s)F(s)\,ds \\ &= \int_0^t m(t-s)F(s)\,ds \\ &= \int_0^t F(t-s)m(s)\,ds \\ &= F * m(t). \end{aligned}$$

(Recall that $m := M'$ is the renewal density.) Consequently, we also have

$$M = F + F * m \tag{2.2}$$

3. Some aspects of the solution H of the renewal equation (1.2) that appear in the preceding example are generally true, as seen in parts (a) and (b) of the following theorem. Part (c) is the "Key Renewal Theorem".

Theorem. *Let the bounded function h be fixed.*

(a) If H satisfies

$$H = h + h * m \tag{3.1}$$

then $\max_{0\le s\le t}|H(s)| < \infty$ for each $t > 0$ (H is said to be "locally bounded") and

$$H = h + H * f. \tag{3.2}$$

(b) Conversely, if H is a locally bounded solution of (3.2) then H also satisfies (3.1).

(c) Suppose that H satisfies (3.1) (or, equivalently, (3.2))) and that $h(t) \ge 0$ for all t and $h(t) \le h(s)$ for all $t_0 \le s \le t$, for some $t_0 > 0$ (h is "eventually decreasing"). Then

$$\lim_{t\to\infty} H(t) = \frac{\int_0^\infty h(u)\,du}{\mu}, \tag{3.3}$$

where, as usual, $\mu = \int_0^\infty t f(t)\,dt$ is the inter-arrival time mean.

The proof of this theorem was sketched in class, using Laplace transforms. I will not repeat it here.

4. Example. The renewal function $M(t) = \mathrm{E}[N(t)]$ satisfies the renewal equation $M = h + M * f$ with the choice $h = F$. Unfortunately, as noted in class, $\int_0^\infty F(t)\,dt = +\infty$, so part (c) of the Theorem is not informative in this case.

5. Key observation. All of our examples rely on a simple but crucial observation. The sequence $\mathbf{W} = \{W_1, W_2, W_3, \ldots\}$ of arrival times of a renewal process splits into the first arrival time W_1 and the (shifted) remaining arrival times $\mathbf{W}' = \{W'_1, W'_2, W'_3, \ldots\}$, where $W'_k := W'_{k+1} - W_1 = X_2 + \cdots + X_{k+1}$. Notice that $\mathbf{W}'$ is independent of W_1 and has the same distribution as $\mathbf{W}$. Another way to say this is that, given that $W_1 = s$, the sequence $\{W_2, W_3, W_4, \ldots\}$ becomes $\{s + W'_1, s + W'_2, s + W'_3, \ldots\}$, and as before $\mathbf{W}' := \{W'_1, W'_2, W'_3, \ldots\}$ has the same distribution as $\mathbf{W}$.

6. Example. Fix $y > 0$ and consider $H(t) := \mathrm{P}[\gamma_t > y]$ for $t \geq 0$. One the one hand, $\gamma_t = X_1 - t$ on the event $\{X_t > t + y\}$. On the event $\{t < X_1 \leq t + y\}$, γ_t cannot be $> y$. Finally, on the event $\{X_1 \leq t\}$ we condition on the value of $X_1 = W_1$ being s, and observe that in this case,

$$\begin{aligned}\gamma_t &= \min\{W_n : W_n > t, n \geq 2\} - t \\ &= \min\{s + W'_k : W'_k > t - s, k \geq 1\} - t \\ &= \min\{W'_k : W'_k > t - s, k \geq 1\} - (t - s) \\ &= \gamma'_{t-s}.\end{aligned}$$

Therefore

$$\mathrm{P}[\gamma_t > y | X_1 = s] = \mathrm{P}[\gamma'_{t-s} > y] = H(t - s),$$

in which the second equality follows because $\gamma'(t - s)$ has the same distribution as γ_{t-s}, being related to $\mathbf{W}'$ in the same was that γ_{t-s} is related to $\mathbf{W}$. By the Law of Total Probability

$$\mathrm{P}[\gamma_t > y, X_1 \leq t] = \int_0^t \mathrm{P}[\gamma_t > t | X_1 = s] f(s)\, ds = \int_0^t H(t - s) f(s)\, ds.$$

It follows that $H(t) = h(t) + H * f(t)$ for $t \geq 0$, where $h(t) = \mathrm{P}[X_1 > t + y] = 1 - F(t + y)$. You have $\int_0^\infty h(t)\, dt = \int_y^\infty [1 - F(u)\, du$, and so by **Theorem 3** above,

$$\lim_{t\to\infty} \mathrm{P}[\gamma_t > y] = \int_y^\infty \frac{1 - F(u)}{\mu}\, du, \qquad y > 0.$$

In other words, the distribution of the random variable γ_t converges, as $t \to \infty$, to that of a random variable γ_∞ with cdf

$$F_{\gamma_\infty}(y) = 1 - \int_y^\infty \frac{1 - F(u)}{\mu}\, du = \int_0^y \frac{1 - F(u)}{\mu}\, du,$$

and density (differentiate!)

$$f_{\gamma_\infty}(y) = \frac{1 - F(y)}{\mu}.$$

7. Example. In this example we take $H(t) = \mu^{-1}\mathrm{E}[\gamma_t]$. On the one hand, because $\gamma_t = W_{N(t)+1} - t$, Wald's Identity tells us that $H(t) = 1 + M(t) - t/\mu$. On the other hand, by the discussion in **Example 6**,

$$\gamma_t = 1_{\{X_1 > t\}}(X_1 - t) + 1_{\{X_t \leq t\}}\gamma'_{t-s}\Big|_{s=W_1}.$$

Therefore, by the Law of Total Probability,

$$H(t) = \mu^{-1}\mathrm{E}[X_1 - t; X_1 > t] + \int_0^t H(t-s)f(s)\,ds. \qquad t \geq 0.$$

That is,

$$H = h + H * f,$$

where

$$h(t) = \frac{1}{\mu}\int_t^\infty (x-t)f(x)\,dx = \frac{1}{\mu}\int_t^\infty [1-F(x)]\,dx,$$

and the second equality above follows from integration by parts. As we saw in class

$$\begin{aligned}
\int_0^\infty h(t)\,dt &= \frac{1}{\mu}\int_0^\infty \int_t^\infty [1-F(x)]\,dx\,dt \\
&= \frac{1}{\mu}\int_0^\infty \int_0^x [1-F(x)]\,dt\,dx \\
&= \frac{1}{\mu}\int_0^\infty x[1-F(x)]\,dx \\
&= \frac{1}{2\mu}\int_0^\infty x^2 f(x)\,dx = \frac{\sigma^2+\mu^2}{2\mu},
\end{aligned}$$

where the penultimate equality results from integration by parts. By **Theorem 3(c)**,

$$\lim_{t\to\infty}\left[M(t) - \frac{t}{\mu}\right] = \frac{\int_0^\infty h(t)\,dt}{\mu} - 1 = \frac{\sigma^2-\mu^2}{2\mu^2}.$$

8. Example. This example stems from the class discussion of the Central Limit Theorem for renewal processes. We define

$$K(t) := \mathrm{E}[N(t)^2]$$

and

$$V(t) := \mathrm{Var}[N(t)] = K(t) - M(t)^2.$$

Notice that $N(t) = 0$ on the event $\{X_1 > t\}$, while on $\{X_1 \leq t\}$ we have $N(t) = 1 + N'(t-s)\Big|_{s=W_1}$, using the logic (and the "prime" notation) of parts **5** and **6**. Therefore

$$N(t)^2 = 1_{\{X_1\leq t\}} + 2\cdot 1_{\{X_1\leq t\}} N'(t-s)\Big|_{s=W_1} + 1_{\{X_1\leq t\}}[N'(t-s)]^2\Big|_{s=W_1},$$

and therefore, for $0 < s \leq t$,

$$\mathrm{E}[N(t)^2 | X_1 = s] = 1 + 2\mathrm{E}[N'(t-s)] + \mathrm{E}[[N'(t-s)]^2] = 1 + 2M(t-s) + K(t-s).$$

It now follows from the Law of Total Probability that

$$K(t) = \mathrm{P}[X_1 \leq t] + \int_0^t [2M(t-s) + K(t-s)]f(s)\,ds,$$

or (what is the same)

$$K(t) = F(t) + (2M + K) * f(t).$$

But $M = F + M * f$, and so $K = F + (2M + K) * f$ means that $K = (2M - F) + K * f$. That is, K satisfies the renewal equation with $h = 2M - F$. It follows that

$$K = M + 2(M * m),$$

and finally that

$$V = M + 2(M * m) - M^2.$$

From this and our knowledge that $M(t) = t/\mu + (\sigma^2 - \mu^2)/2\mu^2 + o(1)$ when the X_k have finite variance σ^2, we can deduce that

$$V(t) = t \cdot \frac{\sigma^2}{\mu^3} + o(t), \qquad t \to \infty.$$