Some 10A Review Problems for 10B

Paul Horn *

In 10B we study integral calculus. As we shall discover, much of 10B will involving finding anti-derivatives. That is, for functions $f(x)$ we will be interested in finding functions $F(x)$ so that

$$\frac{d}{dx}F(x) = f(x).$$

Of course, to do this it is quite important to remember the various derivative rules we learned in the first quarter of calculus (as we will be undoing them.) Here are some reminders and some practice questions to make sure you've got a firm footing on the various derivative rules. Have a question? Is there something important I forgot? Are there typos (answer: almost certainly)? Email me at phorn@math.ucsd.edu! Note that this is not intended to provide a table of basic derivatives. Thankfully, the back of your book has that.

1 A few basic rules:

As notation $f'(x) = \frac{d}{dx}f(x)$. The following rules are our basic building blocks for everything; if f is differentiable, and c is a constant then

$$\begin{aligned} \frac{d}{dx}(f(x) + g(x)) &= f'(x) + g'(x) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x). \\ \frac{d}{dx}cf(x) &= cf'(x) = c\frac{d}{dx}f(x). \end{aligned}$$

Also useful are the following (non-exhaustive) basic derivatives:

$$\begin{array}{ll} \frac{d}{dx}x^n = nx^{n-1} & \frac{d}{dx}c = 0 \\ \frac{d}{dx}\sin(x) = \cos(x) & \frac{d}{dx}\cos(x) = -\sin(x) \\ \frac{d}{dx}e^x = e^x & \frac{d}{dx}\ln(x) = \frac{1}{x} \end{array}.$$

We will use these in future examples!

2 The product rule:

The product rule states that, if f and g are differentiable,

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

Some examples:

Example:

$$\frac{d}{dx}\sin(x)\cos(x) = \cos^2(x) - \sin^2(x).$$

*University of California, San Diego

Example:

$$\frac{d}{dx}e^x \ln(x) = e^x \ln(x) + \frac{e^x}{x}.$$

Example:

$$\begin{aligned} \frac{d}{dx} = x\sin(x)\cos(x) &= \sin(x)\cos(x) + \frac{d}{dx}\sin(x)\cos(x) \\ &= \sin(x)\cos(x) + \cos^2(x) - \sin^2(x). \end{aligned}$$

3 The chain rule:

The chain rule is perhaps the most important and useful of the derivative rules (though saying that may be a bit pretentious.) It is undeniably a rule we will spend quite a bit of time 'undoing' in 10B; the method of substitution we will talk about in this class is really undoing the chain rule.

The chain rule states that

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

In words, the derivative of 'f of something' is the derivative of f, evaluated at that 'something', then multiplied by the derivative of that 'something.' Examples are illuminating:

Example:

$$\frac{d}{dx}\sin(\cos(x)) = \cos(\cos(x)) \cdot (-\sin(x)).$$

Example:

$$\frac{d}{dx}e^{x^2} = e^{x^2} \cdot (2x).$$

Example:

$$\frac{d}{dx}\ln(e^x) = \frac{1}{e^x} \cdot e^x = 1.$$

(This makes sense, as $\ln(e^x) = x$.)

Example:

The chain rule can be used to evaluate the derivative of a^x for constants $a \neq e$. (This rule is given to us in the book, but it's always one that people have trouble to remembering.) Remember for any number a

$$a = e^{\ln a}$$

Remembering that,

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{(\ln a)x} = e^{(\ln a)x}\ln(a) = a^x \ln a.$$

4 The quotient rule:

The quotient rule tells us how to differentiate the ratio of two functions, $f(x)$ and $g(x)$. It states

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

As I learned it, the derivative of 'up over down' is 'down d-up minus up d-down over down down'. However, really, all the quotient rule is is a combination of the chain rule and the product rule:

$$\begin{aligned} \frac{d}{dx}\frac{f(x)}{g(x)} = \frac{d}{dx}f(x)(g(x))^{-1} &= f'(x)(g(x))^{-1} - f(x)(g(x))^{-2}g'(x) \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}. \end{aligned}$$

That's not to say it's not useful. As some examples:

Example:

$$\begin{aligned}\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} &= \frac{\sin(x)\cdot\sin(x) - \cos(x)(-\cos(x))}{\cos^2(x)} \\ &= \frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).\end{aligned}$$

Example:

$$\frac{d}{dx}\frac{\sin(x)}{e^x} = \frac{e^x\cos(x) - \sin(x)e^x}{e^{2x}} = \frac{\cos(x)-\sin(x)}{e^x}.$$

5 Some problems:

Problem: Use the rules above to differentiate the following:

1. $f(x) = \sin^2(x)\cos(x)$.
2. $f(x) = e^{\sin(x)}$.
3. $f(x) = 4^{-\sin(x)\cos(x)}$.
4. $f(x) = (\sqrt{e^{\cos(x)}} - 2)^2$.
5. $f(x) = \frac{\sec(x)+\tan(x)}{x^2}$.

Problem: Suppose we know that $\frac{d}{dx}x = 1$, but not $\frac{d}{dx}x^n = nx^{n-1}$. (Unrealistic, I know.) Why does the fact that $x^n = nx^{n-1}$ follow from the product rule?

Problem: Find $\frac{d}{dx}e^{x\ln x}$.

Problem: Find $\frac{d}{dx}x^x$. *(Hint: It's not $x(x^{x-1})$), that rule only works when the exponent is constant.) (Hint 2: The fact that $x = e^{\ln x}$ is often useful.)*

If you have any good problems for me, let me know!