Sample Midterm Exam

Math 20F
8/22/08

Name: ______________________________
Section: ______________________________

Read all of the following information before starting the exam:

- READ EACH OF THE PROBLEMS OF THE EXAM CAREFULLY!
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- A single 8 1/2 $\times$ 11 sheet of notes (double sided) is allowed. No calculators are permitted.
- Circle or otherwise indicate your final answers.
- Please keep your written answers clear, concise and to the poin.
- This test has xxx problems and is worth xxx points. It is your responsibility to make sure that you have all of the pages!
- Turn off cellphones, etc.
- Good luck!

1	
2	
3	
4	
5	
$\sum$	

1. (*0 points*)

(**a**) Let $A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \\ 1 & 2 & 2 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Solve $A\mathbf{x} = b$.

Answer: By row reduction:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}.$$

.

(**b**) Suppose $T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find the matrix for the linear transformation T.

Answer: Note that the hypothesis imply $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, so

$$A = \begin{pmatrix} 2 & 0 \\ 2 & -1 \end{pmatrix}.$$

. (**c**) Define linear independence.

Answer: Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent if $\alpha_1\mathbf{v}_1 + \ldots + \alpha_n\mathbf{v}_n = \mathbf{0}$ implies that $\alpha_i = 0$ for all i. (That is, the only linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ giving the $\mathbf{0}$ vector is the trivial one taking all coefficients to be zero.)

(**d**) Let $A = \begin{pmatrix} 1 & 2 & -2 \\ 3 & 4 & 0 \\ 2 & 5 & -7 \end{pmatrix}$. Are the columns of A linearly independent?

Answer: Row reducing A we get that

$$A \sim \begin{pmatrix} 1 & 2 & -2 \\ 0 & -2 & 6 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since not every column is a pivot column, the columns are not linearly independent.

2. (*0 points*)

(**a**) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Let $B = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & -1 \end{pmatrix}$. Compute A^T, AB and $B^T - 3A$.

Answer:

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad AB = \begin{pmatrix} 1 & 7 & 2 \\ 0 & 3 & 3 \\ 1 & 6 & 1 \end{pmatrix}, \quad B^T - 3A = \begin{pmatrix} -1 & -9 & -4 \\ 3 & -8 & 2 \\ -2 & -2 & -4 \end{pmatrix}.$$

(**b**) Let $A = \begin{pmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & 0 \end{pmatrix}$. Compute A^{-1}.

Answer:

$$A^{-1} = \frac{1}{4}\begin{pmatrix} -18 & 6 & 1 \\ -6 & 2 & 1 \\ 10 & -2 & 1 \end{pmatrix}.$$

(c) Suppose $A\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$, and $A\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$. Find A^{-1}.

Answer: Note that the hypothesis imply that $A^{-1}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix}$ and $A^{-1}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$, so

$$A^{-1}=\begin{pmatrix}1 & 1\\ -2 & 2\end{pmatrix}.$$

3. (*0 points*)

(a) Suppose $A=\begin{pmatrix}0&0&0&0&1\\0&0&0&-2&1\\0&0&-3&0&1\\0&1&2&2&-1\\3&0&0&0&2\end{pmatrix}$. Compute $\det(A)$.

Answer: Via 2 row exchanges,

$$A\sim A'=\begin{pmatrix}3&0&0&0&2\\0&1&2&2&-1\\0&0&-3&0&1\\0&0&0&-2&1\\0&0&0&0&1\end{pmatrix}$$

Thus $\det(A)=(-1)^2\det(A')$. Since A' is upper triangular this is easy to compute, and we get $\det(A)=18$.

(b) Suppose $A=\begin{pmatrix}1&-1\\2&1\end{pmatrix}$, and $\mathbf{b}=\begin{pmatrix}2\\1\end{pmatrix}$. Use Cramer's rule to solve $A\mathbf{x}=\mathbf{b}$.

Answer: Note that:

$$\begin{aligned}\det(A_1(\mathbf{b})) &= \det\begin{pmatrix}2&-1\\2&1\end{pmatrix}=3\\ \det(A_2(\mathbf{b})) &= \det\begin{pmatrix}1&2\\2&1\end{pmatrix}=-3\\ \det(A) &= 3\end{aligned}$$

So Cramer's rule gives $\mathbf{x}=\begin{pmatrix}1\\-1\end{pmatrix}$.

(c)

Find the area of the *triangle* with vertices $(2,3)$, $(4,7)$ and $(8,4)$ using the methods of this course.

Answer: The triangle is determined by the vectors $(2,4)$ and $(6,1)$, so

$$Area=\frac{1}{2}\left|\det\begin{pmatrix}2&6\\4&1\end{pmatrix}\right|=11.$$

4. (*0 points*) For each statement, mark it true of false. If it is false give a (counter)example or brief proof. If it is true give a reason - if the reason is a theorem, state the theorem, otherwise give a brief proof. No credit for answers without a correct reason or example. Unless explicitly noted, there are no condition on the dimensions of matrices A and B.

(**a**) If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ are linearly dependent, then one of the vectors is a multiple of another.

FALSE: Take $\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{v_3} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(**b**) If $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$, the columns of A span $\mathbf{R}^m$.

TRUE: If $A\mathbf{x} = \mathbf{b}$ is consistent for all then for any $\lfloor$, $\lfloor$ can be written as a linear combination of the columns of A (with coefficients given by $\mathbf{x}$.) This is exactly what it means to span $\mathbf{R}^m$.

(**c**) If $AB = AC$, then $B = C$.

FALSE: If A is the zero matrix, A times any matrix B is the zero matrix, so this is certainly false.

(**d**) If the transformation $T(\mathbf{x}) = AB\mathbf{x}$ is onto, then the transformation $T'(x) = A\mathbf{x}$ is onto

TRUE: We need to check that for each $\mathbf{b}$, there is a $\mathbf{y}$ such that $\mathbf{b} = A\mathbf{y}$. But, we know that for every $\mathbf{b}$ we can write $\mathbf{b} = AB\mathbf{x}$ for some $\mathbf{x}$. We simply can take $\mathbf{y} = B\mathbf{x}$. This checks the definition of onto.

(**e**) If A and B are $n \times n$, and invertible, then $A + B$ is invertible.

FALSE: If A is invertible, so is $-A$, and $A + (-A)$ is the zero matrix which is not invertible.

(**f**) If A is row equivalent to B, then $\det(A) = \det(B)$.

FALSE: Though $\det(A)$ is unchanged by row replacement, swapping rows and multiplying a row by a scalar changes the determinant. Example:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

but the determinants are different.

(**g**) If the columns of A are linearly dependent, then $\text{Col}(A)$ is not a vector space.

FALSE: $\text{Col}(A)$ is the span of the columns of A, and the span of any vectors is a vector space (even if they are linearly dependent.

5. (*0 points*)

(**a**) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation verify that the range of $T = \{\mathbf{v} \in \mathbb{R}^m : T(\mathbf{x}) = \mathbf{v} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$, is a vector space. (Note the range is not necessarily all of $\mathbb{R}^m$. $\mathbb{R}^m$ is the co-domain, not the range.).

Answer: It suffices to check that the range is a subspace as it sits inside $\mathbb{R}^m$. Actually this is what I intended to ask. Let $\mathcal{R}$ denote the range. If $\mathbf{u}$, $\mathbf{v} \in \mathcal{R}$ then by definition of $\mathcal{R}$, $\mathbf{u} = T(\mathbf{x})$ and $\mathbf{v} = T(\mathbf{y})$. Note that

$$\mathbf{u} + \mathbf{v} = T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$$

so $\mathbf{u} + \mathbf{v} \in \mathcal{R}$.

If $\mathbf{u} = T(\mathbf{x}) \in \mathcal{R}$ then $c\mathbf{u} = cT(\mathbf{x}) = T(c\mathbf{x}) \in \mathcal{R}$. Finally, we need to check $\mathbf{0} \in \mathcal{R}$, but $T(\mathbf{0}) = \mathbf{0}$ so indeed it is. We've checked the three conditions so $\mathcal{R}$ is a subspace of $\mathbb{R}^m$.

(**b**) Verify that the derivative operator $\frac{d}{dx}$ is a linear transformation from the vector space of polynomials of degree at most 3 to the vector space of polynomials of degree at most 2. Is this linear transformation $1-1$? Onto?

Answer: Here we use elementary properties of the derivative. Recall a linear transformation satisfies: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c\mathbf{u}) = cT(\mathbf{v})$. In our case, $T = \frac{d}{dx}$ and our vectors $\mathbf{u}$ and $\mathbf{v}$ are polynomials $f(x)$ and $g(x)$. Note:

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

so the first property holds, and

$$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$$

so the second property also holds, and hence $\frac{d}{dx}$ is a linear transformation.
$\frac{d}{dx}1 = \frac{d}{dx}2 = 0$, so $\frac{d}{dx}$ is not 1-1, however $\frac{d}{dx}\int f(x)dx = f(x)$, so $\frac{d}{dx}$ is onto.