Midterm Exam

Math 20F
8/22/08

Name: ____________________
Section: ____________________

Read all of the following information before starting the exam:

- READ EACH OF THE PROBLEMS OF THE EXAM CAREFULLY!
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- A single 8 1/2 $\times$ 11 sheet of notes (double sided) is allowed. No calculators are permitted.
- Circle or otherwise indicate your final answers.
- Please keep your written answers clear, concise and to the poin.
- This test has 5 problems and is worth 100 points.. It is your responsibility to make sure that you have all of the pages!
- Turn off cellphones, etc.
- Good luck!

1	
2	
3	
4	
5	
$\sum$	

1. (*25 points*)

a. (*5 pts*) Solve the system of equations:

$$\begin{aligned} 2x_1 + 0x_2 + 3x_3 &= 5 \\ x_1 + x_2 + x_3 &= 2 \\ x_1 - 2x_2 + 2x_3 &= 4 \end{aligned}$$

Answer: Note (swapping equations 1 & 2 to make things a bit cleaner) we get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 0 & 3 & 5 \\ 1 & -2 & 2 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & 1 \\ 0 & -3 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

so $\mathbf{x} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$.

b. (*10 pts*) Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates points about the origin $\pi/2$ radians *clockwise*. Compute $T\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Answer: Note: $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Also,

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

c. (*10 pts*) Let $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 4 \end{pmatrix}$. Find the solution set for $A\mathbf{x} = 0$. In other words, find Nul(A).

Answer Note

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -5 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -7 & 5 \\ 0 & 1 & 5 & -2 \end{pmatrix}.$$

This gives $x_1 = 7x_3 - 5x_4$ and $x_2 = -5x_3 + 2x_4$, so

$$Nul(A) = \left\{ \begin{pmatrix} 7x_3 - 5x_4 \\ -5x_3 + 2x_4 \\ x_3 \\ x_4 \end{pmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \text{Span}\left(\begin{pmatrix} 7 \\ -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right).$$

2. (*20 points*)

a. (*6 pts*) Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix}$. Compute $C = A^2 - B(A^T)$. Is C invertible?

Answer: Note

$$A^2 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \quad B(A^T) = \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 6 & 3 \end{pmatrix},$$

so

$$C = A^2 - B(A^T) = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 6 & 6 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 0 & 0 \end{pmatrix}.$$

Note $\det(C) = 0$ so C is *not* invertible.

b. (*4 pts*) Suppose $A^{-1} = \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.

Answer: Note that

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

c. (*10 pts*) Let $A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \\ 0 & 2 & 5 \end{pmatrix}$. Compute A^{-1}.

Answer: Note

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 17 & -8 & -4 \\ 0 & -1 & 0 & -10 & 5 & 2 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -13 & 7 & 2 \\ 0 & 1 & 0 & 10 & -5 & -2 \\ 0 & 0 & 1 & -4 & 2 & 1 \end{array}\right).$$

so $A^{-1} = \begin{pmatrix} -13 & 7 & 2 \\ 10 & -5 & -2 \\ -4 & 2 & 1 \end{pmatrix}$.

3. (*20 points*)

a. (*10 pts*) Suppose

$$A = LU = \begin{pmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Compute $\det(A)$. *Answer:* Note that since L and U are triangular, their determinant is the product of their diagonal entries.

$$\det(A) = \det(LU) = \det(L)\det(U) = -9 \cdot -2 = 18.$$

b. (*10 pts*) Find the volume of the parallelepiped determined by the 3 vectors $\mathbf{v_1} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$, and $\mathbf{v_3} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Answer:

$$V = \left| \det \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 4 & 0 & -1 \end{pmatrix} \right| = \left| 3 \det \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} \right| = |3 \cdot (-5)| = 15.$$

4. (*20 points*) For each statement, mark it true of false. If it is false give a (counter)example or brief (dis)proof. If it is true give a reason - if the reason is a theorem, state the theorem, otherwise give a brief proof. No credit for answers without a correct reason or example. Unless explicitly stated, no assumptions are made on the dimensions, etc., of matrices.

a. (*5 pts*) If the transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto, then the columns of A are linearly independent.

FALSE: As an example, suppose $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Every row is a pivot row, so the transformation is onto, but the columns are linearly dependent.

b. (*5 pts*) If A and B are square matrices, and AB is invertible, then A is invertible.

TRUE: Note that $AB(AB)^{-1} = I$. This means that $A(B(AB)^{-1}) = I$. Thus A has a right inverse. But the Invertible Matrix Theorem states that if A has a right inverse, A is invertible. As an alternate proof, note that AB being invertible means $\det(AB) \neq 0$. But $\det(AB) = \det(A)\det(B)$, so in particular $\det(A) \neq 0$ so A is invertible.

WARNING: There is a theorem saying that 'If A and B are invertible, then AB is invertible, with $(AB)^{-1} = B^{-1}A^{-1}$. We can't apply this here! We don't know that A is invertible (that's what we're trying to show!) so the hypothesis don't hold, and we can't say that $(AB)^{-1} = B^{-1}A^{-1}$ - we don't even know a priori that they exist. If A and B are square, it turns out that they do - but this requires a proof like the above. (A and B need not be square for AB to be invertible, so it is not in general true that if AB is invertible A is invertible, even!)

c. (*5 pts*) If AB and BA both exist (are well defined products), then A and B are both $n \times n$ (square).
FALSE: If A is $m \times n$, and B is $n \times m$ (not necessarily square!) then AB and BA both exist. AB is $m \times m$ and BA is $n \times n$.

d. (*5 pts*) If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.
TRUE: If A is invertible, then $AA^{-1} = I$. Note:

$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}).$$

Rearranging, we see $\det(A^{-1}) = \frac{1}{\det(A)}$.

5. (*15 points*) **a.** (*10 pts*) Let $\mathbb{P}_3$ denote the vector space of polynomials of degree at most 3. Verify that $\mathcal{I} = \{f(x) \in \mathbb{P}_3 : \int_{-1}^{1} f(x)dx = 0\}$ is a subspace of $\mathbb{P}_3$.
Answer: We need to check that $\mathcal{I}$ is closed under addition and scalar multiplication and that $0 \in \mathcal{I}$. Since $\int_{-1}^{1} 0dx = 0$, $0 \in \mathcal{I}$.
For any two polynomials $f(x), g(x) \in \mathcal{I}$:

$$\int_{-1}^{1} f(x) + g(x)dx = \int_{-1}^{1} f(x)dx + \int_{-1}^{1} g(x)dx = 0 + 0 = 0,$$

so $f(x) + g(x) \in \mathcal{I}$.
Finally, we need to show that if $f(x) \in \mathcal{I}$ that $cf(x) \in \mathcal{I}$. But,

$$\int_{-1}^{1} cf(x)dx = c\int_{-1}^{1} f(x)dx = c \cdot 0 = 0,$$

so $cf(x) \in \mathcal{I}$ as desired. Thus $\mathcal{I}$ is a subspace.

b. (*5 pts*) Is $\mathcal{I}' = \{f(x) \in \mathbb{P}_3 : \int_{-1}^{1} f(x)dx = 1\}$ a subspace of $\mathbb{P}_3$? Why or why not?
Answer: Note that $\int_{-1}^{1} 0dx \neq 1$ so $0 \notin \mathcal{I}'$ so $\mathcal{I}'$ is not a subspace. (Also, one can check that $\mathcal{I}'$ is not closed under addition or scalar multiplication, so it really fails on all counts!)

(**c**) *The bonus question that almost was*: I almost included this as a bonus question but removed it because it was a bit too hard and required a dimension argument (which wasn't being tested to answer). I like the question though: Find a basis for $\mathcal{I}$ from part (a).
Answer: It is easy to check that $\int_{-1}^{1} xdx = 0$ and $\int_{-1}^{1} x^3dx = 0$. Slightly harder to find is that $\int_{-1}^{1} x^2 - \frac{1}{3}dx = 0$. This gives 3 linearly independent vectors in $\mathcal{I}$:

$$\mathcal{B} = \left\{x, x^2 - \frac{1}{3}, x^3\right\}.$$

Why must this be a basis? Well, not every function in $\mathbb{P}^3$ is in $\mathcal{I}$, so $\dim(\mathcal{I}) < \dim(\mathbb{P}^3)$. Since $\dim(\mathbb{P}^3) = 4$, this says $\dim(\mathcal{I}) \leq 3$. Since I have 3 linearly independent vectors in $\mathcal{I}$, $\mathcal{B}$ *must* be a basis.

Scrap Page
(please do not remove this page from the test packet)