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Cheeger's Inequality for Directed Graphs

1 The Perron Vector

Previously we looked at Cheeger's Inequality as it relates to undirected graphs. We now prove that a similar result holds for directed graphs as well. As before, we define the transition probability matrix for G, a strongly connected directed graph, as

$$P(x,y) = \begin{cases} \frac{1}{d_x} & \text{if } (x,y) \text{ is an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

where d_x is the out-degree of x.

The Perron-Frobenius Theorem states that there exists a unique left eigenvector ϕ such that $\phi(v) > 0$ for all v and $\phi P = \rho\phi$, where ρ is the spectral radius. Since $P1 = 1$ it follows that $\rho = 1$ and the Perron-Frobenius Theorem shows that all eigenvalues have absolute value bounded by 1. We scale ϕ so that

$$\sum_v \phi(v) = 1,$$

and call this ϕ the Perron vector. Notice that whenever G is strongly connected and aperiodic, the stationary distribution is exactly the Perron vector.

Example 1. The graph G is regular.
We say a directed graph G is regular if all out-degrees are equal. If $|V(G)| = n$ then

$$\phi = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}).$$

Example 2. The graph G is Eularian.
We say a directed graph is Eularian if at every vertex the in-degree equals the out-degree. Then

$$\phi = \left(\frac{d_x}{\sum_y d_y} \right)_{x \in V(G)}.$$

Exercise 1. Find the Perron vector, ϕ, for the graph shown in Figure 1.

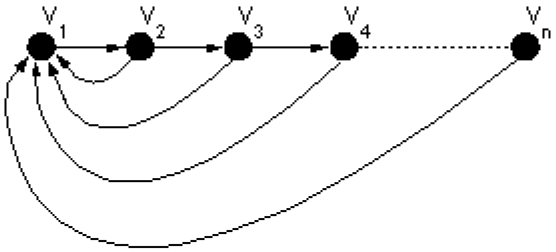

Figure 1: Graph for Exercise 1

2 Circulation of a Directed Graph

We now introduce a new definition to assist with writing Cheeger's inequality for G.

Definition. A function $F : E(G) \to \mathbb{R}^+ \cup \{0\}$ is a *circulation* if for every v,

$$\sum_{\substack{u \\ u \to v}} F(u,v) = \sum_{\substack{w \\ v \to w}} F(v,w).$$

As an example, if the graph contains a cycle, then a function which is constant on the edges of that cycle and zero otherwise is a circulation. This is slightly different from usual network flows which have sources and sinks, for example electrical networks. The circulation as defined here has no sources or sinks.

Definition. Let ϕ be the Perron vector of G and let P be the transition probability vector of G. Then let

$$F_\phi(u,v) = \phi(v)P(u,v).$$

Fact. F_ϕ is a circulation.
Proof. Fix v, then

$$\begin{aligned} \sum_{\substack{u \\ u \to v}} F_\phi(u,v) &= \sum_{\substack{u \\ u \to v}} \phi(u)P(u,v) \\ &= \phi P(v) \\ &= \phi(v) \\ &= \sum_{\substack{w \\ v \to w}} \phi(v)P(v,w) \\ &= \sum_{\substack{w \\ v \to w}} F_\phi(v,w). \end{aligned}$$

□

Note that a reversible Markov Chain will satisfy $F_\phi(u,v) = F_\phi(v,u)$.

Definition. Let $S \subseteq V(G)$. Define the *out-boundary* of S, ∂S, as

$$\partial S = \{(u,v) \in E(G) | u \in S, v \notin S\}.$$

Note, for directed graphs in-boundary and out-boundary can be quite different. However we do have the following.

Exercise 2. Prove that if F is a circulation then $F(\partial S) = F(\partial \overline{S})$. (Note ∂S is the out-boundary and $\partial \overline{S}$ is the in-boundary.)

Let $S \subseteq V(G)$ and F a circulation and define the following:

$$\begin{aligned} F(v) &= \sum_{\substack{w \\ v \to w}} F(v,w), \\ F(S) &= \sum_{v \in S} F(v), \\ h(S) &= \frac{F_\phi(\partial S)}{\min\{F_\phi(S), F_\phi(\overline{S})\}}, \\ h_G &= \min_S h(S). \end{aligned}$$

3 The Raleigh Quotient and Cheeger Inequality

The goal is to have eigenvalues that help us with the Cheeger constant so we will work backwards to get what we want. We start by defining the Raleigh quotient.

Definition. Let $f : V(G) \to \mathbb{C}$ and P be the transition probability matrix. Then we define the *Raleigh quotient*, $R(f)$, as

$$R(f) = \frac{\sum_{u \to v} |f(u) - f(v)|^2 \phi(u) P(u,v)}{\sum_v |f(v)|^2 \phi(v)}.$$

Fact. This is equivalent to writing

$$R(f) = 2 \frac{\langle fL, f \rangle}{\langle f\Phi, f \rangle}$$

where

$$L = \Phi - \frac{\Phi P + P^* \Phi}{2}$$

with

$$\Phi = \begin{pmatrix} \phi(v_1) & & 0 \\ & \ddots & \\ 0 & & \phi(v_n) \end{pmatrix}.$$

Proof.

$$\begin{aligned} R(f) &= \frac{\sum_{u\to v}(f(u)-f(v))(\overline{f}(u)-\overline{f}(v))\phi(u)P(u,v)}{f\Phi f^*} \\ &= \frac{2f\Phi f^* - \sum_{u\to v}(f(u)\phi(u)P(u,v)\overline{f}(v)+\overline{f}(u)\phi(u)P(u,v)f(v))}{f\Phi f^*} \\ &= 2 - \frac{f\Phi P f^* + \overline{f}P^*\Phi f^*}{f\Phi f^*} \\ &= \frac{f(2\Phi - \Phi P - P^*\Phi)f^*}{f\Phi f^*} \\ &= 2\frac{fLf^*}{f\Phi f^*} \end{aligned}$$

as desired. □

Now we define the *Laplacian*, $\mathcal{L}$, as

$$\mathcal{L} = I - \frac{\Phi^{1/2}P\Phi^{-1/2} + \Phi^{-1/2}P^*\Phi^{1/2}}{2} = \Phi^{-1/2}L\Phi^{-1/2}$$

Fact. If we let $g = f\Phi^{1/2}$, then $R(f) = 2\frac{\langle fL, f\rangle}{\langle f\Phi, f\rangle} = 2\frac{\langle g\mathcal{L}, g\rangle}{\langle g, g\rangle}$.
Proof. Let $g = f\Phi^{1/2}$, then

$$\begin{aligned} \langle fL, f\rangle &= fLf^* \\ &= g\Phi^{-1/2}L\Phi^{-1/2}g^* \\ &= g\mathcal{L}g^*. \end{aligned}$$

It follows that if $\mathcal{L}$ has eigenvalues $\lambda_0 = 0, \lambda_1, \ldots$, then

$$\begin{aligned} \lambda_1 &= \inf_{\substack{f \\ \Sigma_x f(x)\phi(x)=0}} \frac{R(f)}{2} \\ &= \inf_{\substack{g \\ \langle g, \Phi^{1/2}1\rangle = 0}} \frac{\langle g\mathcal{L}, g\rangle}{\|g\|^2}. \end{aligned}$$

We can then write *Cheeger's Inequality*,

$$2h(G) \geq \lambda_1 \geq \frac{h^2(G)}{2}.$$

The proof of this theorem will follow in a later discussion.

References

[1] Fan Chung, Laplacians and the Cheeger inequality for directed graphs, http://www.math.ucsd.edu/∼fan/wp/dichee.pdf