

Conceptually, the reciprocal of the spectrum in the expression for $\Psi(e^{-i\lambda})$ in Example 6.5.1 plays the role of Γ_k^{-1} in the Yule-Walker equations, as $k \rightarrow \infty$. This relationship motivates the general definition of inverse autocovariances, which appear above.

Definition 6.5.1 Inverse Autocovariance For a weakly stationary time series $\{X_t\}$ with positive spectral density f , the inverse autocovariance function is a sequence $\{\tilde{\gamma}(k)\}$ that satisfies

$$\sum_{k \in \mathbb{Z}} \gamma(k) \tilde{\gamma}(j - k) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{else.} \end{cases}$$

The inverse autocorrelation function $\{\tilde{\rho}(k)\}$ is defined via $\tilde{\rho}(k) = \tilde{\gamma}(k)/\tilde{\gamma}(0)$.

Definition 6.5.1 is heuristically like the equation $\Gamma_n \Gamma_n^{-1} = 1_n$, where the entries of Γ_n^{-1} are approximately given by the $\tilde{\gamma}(j - k)$. Example 6.5.1 indicates that inverse autocovariances can be related to the reciprocal spectral density; this is proved in the following results.

Proposition 6.5.1 *If $\{X_t\}$ is a weakly stationary invertible time series with spectral density f , which is strictly positive, then the inverse autocovariance is given by*

$$\tilde{\gamma}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{f(\lambda)} e^{ik\lambda} d\lambda.$$

Proof of Proposition 6.5.1. The result is verified by a calculation:

$$\begin{aligned} & \sum_{k \in \mathbb{Z}} \gamma(k) \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{f(\lambda)} e^{i(j-k)\lambda} d\lambda \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sum_{k \in \mathbb{Z}} \gamma(k) e^{-ik\lambda}}{f(\lambda)} e^{ij\lambda} d\lambda \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij\lambda} d\lambda, \end{aligned}$$

which equals zero unless $j = 0$, in which case the integral is unity. \square

Remark 6.5.1 Theorem 6.7.1, given below, shows another way of viewing the relationship of Proposition 6.5.1: the spectral density f evaluated at various Fourier frequencies $\pi j/k$ for $j = 0, 1, \dots, k - 1$ approximately yields the eigenvalues of Γ_k ; in addition, the reciprocal spectral density $1/f$ evaluated at the same Fourier frequencies approximately yields the eigenvalues of Γ_k^{-1} .

To compute the inverse autocovariances for a given ARMA time series $\{X_t\}$, we can use the techniques introduced above – so long as spectral density is positive. This positivity is equivalent to the process being invertible.

Example 6.5.2 The Inverse Autocovariance of an MA(1) Consider an MA(1) process with parameter $\theta_1 \in (-1, 1)$ and innovation variance σ^2 . Thus f is positive, and

$$f^{-1}(\lambda) = \sigma^{-2} |1 + \theta_1 e^{-i\lambda}|^{-2}.$$

This is recognized as the spectral density of an AR(1) process, of parameter $\phi_1 = -\theta_1$ and innovation variance σ^{-2} . Hence

$$\tilde{\gamma}(k) = \frac{(-\theta_1)^{|k|}}{1 + \theta_1^2} \sigma^{-2}.$$

Generalizing the method of Example 6.5.2, for an invertible ARMA process satisfying $\phi(B)X_t = \theta(B)Z_t$ for $\{Z_t\}$ a white noise of variance σ^2 , we have $f(\lambda) = \sigma^2 |\theta(e^{-i\lambda})|^2 |\phi(e^{-i\lambda})|^{-2}$. Then $1/f(\lambda) = \sigma^{-2} |\phi(e^{-i\lambda})|^2 |\theta(e^{-i\lambda})|^{-2}$, which is the spectral density of an ARMA process $\{Y_t\}$ satisfying $\theta(B)Y_t = \phi(B)W_t$, where $\{W_t\}$ is a white noise of variance σ^{-2} . One caution about this approach, is that the plus and minus conventions of MA and AR polynomials have been reversed. Here we now treat $\phi(B)$ as a moving average polynomial, although it has the structure $\phi(B) = 1 - \sum_{j=1}^p \phi_j B^j$, i.e., the coefficients are preceded by minus signs, whereas a moving average polynomial is typically written with plus signs before the coefficients.

Example 6.5.3 Optimal Interpolation Suppose that we have a full time series $\{X_t\}$ available to us, except that some of the values are missing. In practice this may arise due to faulty machinery (where a recording device malfunctions, resulting in a missing observation) or due to human error (in a business survey, late reporting or nonsensical values – due to a typing mistake – produce missing values). Let us suppose that X_0 is missing, but the rest of the time series is available, and we would like to compute the optimal interpolation. We claim that the solution is

$$P_{\text{sp}\{X_j, j \neq 0\}} X_0 = - \sum_{j \neq 0} \tilde{\rho}(j) X_j.$$

In order for the expression on the right hand side to be the projection, it is sufficient that the error is orthogonal to every X_k for $k \neq 0$. The error is

	MA(q)	AR(p)	ARMA(p, q)
$\rho(k)$	0 if $k > q$	geometric decay	geometric decay
$\alpha(k)$	geometric decay	0 if $k > p$	geometric decay
$\tilde{\rho}(k)$	geometric decay	0 if $k > p$	geometric decay

Table 6.1: Behavior of autocorrelations $\rho(k)$, partial autocorrelations $\alpha(k)$, and inverse correlations $\tilde{\rho}(k)$ as $k \rightarrow \infty$, for MA(q), AR(p), and ARMA(p, q) processes.

$X_0 + \sum_{j \neq 0} \tilde{\rho}(j) X_j$, and its covariance with any X_k is

$$\gamma(k) + \sum_{j \neq 0} \tilde{\rho}(j) \gamma(j - k) = \sum_{j \in \mathbb{Z}} \tilde{\rho}(j) \gamma(j - k) = \sum_{j \in \mathbb{Z}} \tilde{\gamma}(j) \gamma(j - k) / \tilde{\gamma}(0),$$

which is zero for $k \neq 0$ by the definition of inverse autocovariance.

6.6 Properties of Correlations

We've seen that the autocorrelation sequence of an MA(q) is zero for lags exceeding q , and by Example 6.4.1 that the partial autocorrelation sequence of an AR(p) is zero for lags exceeding p . The same is true of the inverse autocorrelations of an AR(p), because its inverse autocorrelations are the same as the autocorrelations of a corresponding MA(p) process. Also, Proposition 5.7.1 states that ARMA autocovariances have geometric decay; this also applies to inverse autocorrelations, and can likewise be deduced for partial autocorrelations – see the proof of Proposition 6.4.1. These results are summarized in Table 6.1.

We say that a sequence has geometric decay if (5.7.4) holds. One application of Table 6.1 is to model identification: if the sample estimates of autocorrelations, partial autocorrelations, or inverse correlations are observed to have behavior given by one of the columns, then it lends evidence to the hypothesis that the data is generated by that model.

Example 6.6.1 MA(q) Identification Suppose that sample estimates of the autocovariance function appear to be approximately zero for all lags greater than three, while the sample partial autocorrelations and inverse autocorrelations have geometric decay. Then we might posit an MA(3) model for the data; if the data was from an MA(q) process with $q > 3$, we'd expect that the fourth and higher lags of the sample autocorrelation to be nonzero. Conversely, if the data was generated from an MA(2) process, then the third lag of the sample autocorrelation should be approximately zero.

6.7 Eigenvalues of Covariance Matrices

The autocovariances are related to the spectral density by (6.1.3), but there is a further approximation that is extremely useful. In many applications, such as computing the Gaussian likelihood or forecast formulas, it is necessary to compute a Toeplitz matrix of autocovariances; this can be approximated in terms of the spectral density and a unitary matrix. Recall that the Toeplitz covariance matrix of dimension n is defined via

$$\Gamma_n = \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) \\ \gamma(2) & \gamma(1) & \cdots & \gamma(n-3) \\ \vdots & \vdots & \vdots & \vdots \\ \gamma(n-1) & \gamma(n-2) & \cdots & \gamma(0) \end{bmatrix}.$$

The matrix is non-negative definite, and has an eigenvalue decomposition. Moreover, the Toeplitz structure implies a specific form for the eigenvalue decomposition, in an approximate sense. Recall that the conjugate transpose of a matrix A is denoted by A^* .

Theorem 6.7.1 Eigen-decomposition of Toeplitz Matrices *If Γ_n is the Toeplitz covariance matrix of a weakly stationary time series with spectral density f , then for large n*

$$\Gamma_n \approx Q \Lambda Q^*, \quad (6.7.1)$$

where Q has complex entries and is unitary (i.e., $Q^* = Q^{-1}$), but Λ is diagonal and real-valued. Let $\lambda_\ell = \pi(\ell-1)/n$ for $1 \leq \ell \leq n$, which are called the Fourier frequencies; the ℓ th diagonal entry of Λ is $f(\lambda_\ell)$, and the jk th entry of Q is $Q_{jk} = \exp\{i(j-1)\lambda_k\}/\sqrt{n}$.

Sketch of Proof of Theorem 6.7.1. We do not prove this result, but provide a heuristic derivation instead. The jk th entry of $Q \Lambda Q^*$ is

$$\begin{aligned} [Q \Lambda Q^*]_{jk} &= \sum_{\ell=1}^n Q_{j\ell} \Lambda_{\ell\ell} \bar{Q}_{k\ell} \\ &= n^{-1} \sum_{\ell=1}^n e^{i(j-1)\lambda_\ell} f(\lambda_\ell) e^{-i(k-1)\lambda_\ell} \\ &= n^{-1} \sum_{\ell=0}^{n-1} f(\pi\ell/n) e^{i(j-k)\pi\ell/n} \\ &\approx \pi^{-1} \int_0^\pi f(\omega) e^{i(j-k)\omega} d\omega = \gamma(j-k). \end{aligned}$$

The final approximation follows from Riemann integration. This shows the approximation (6.7.1) holds entry-by-entry, and is a matrix analogue of the formula $\gamma(h) = (2\pi)^{-1} \int_{-\pi}^\pi f(\lambda) e^{ih\lambda} d\lambda$. \square

Remark 6.7.1 Inverting the relationship (6.7.1) yields

$$\Lambda \approx Q^* \Gamma_n Q. \quad (6.7.2)$$

This forms the matrix analogue of the formula $f(\lambda) = \sum_{h=-\infty}^\infty \gamma(h) e^{-ih\lambda}$.

Remark 6.7.2 As another consequence of Theorem 6.7.1, if f is strictly positive, we have

$$\Gamma_n^{-1} \approx Q \Lambda^{-1} Q^*,$$

with Λ^{-1} a diagonal with entries given by the reciprocals of f evaluated at the Fourier frequencies.

Example 6.7.1 Suppose we have two autocovariance functions γ_x and γ_y with associated $n \times n$ Toeplitz covariance matrices Γ_n^x and Γ_n^y . Then applying Theorem 6.7.1 yields

$$\Gamma_n^x \Gamma_n^y \approx Q \Lambda_x \Lambda_y Q^*, \quad (6.7.3)$$

where Λ_x and Λ_y are diagonal matrices of Fourier frequencies corresponding to the spectral densities of γ_x and γ_y respectively. Viewing matrix multiplication as an approximate convolution, the above relationship is a matrix version of the formula

$$\sum_{k=-\infty}^\infty \gamma_x(h-k) \gamma_y(k) = \frac{1}{2\pi} \int_{-\pi}^\pi f_x(\lambda) f_y(\lambda) e^{ih\lambda} d\lambda. \quad (6.7.4)$$