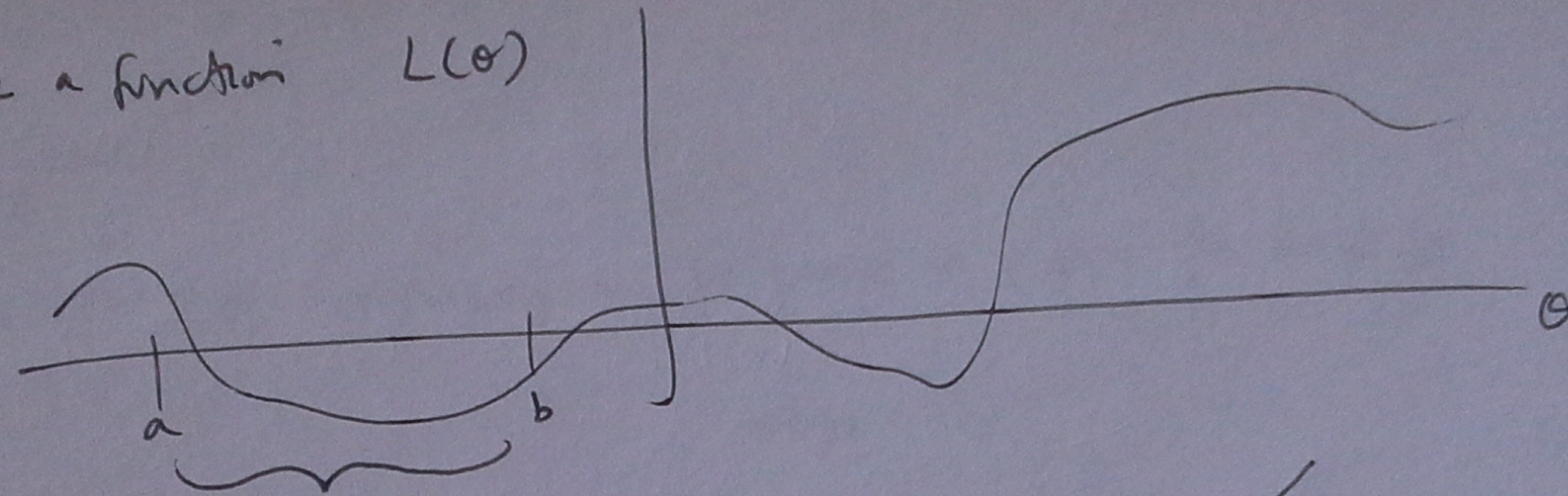
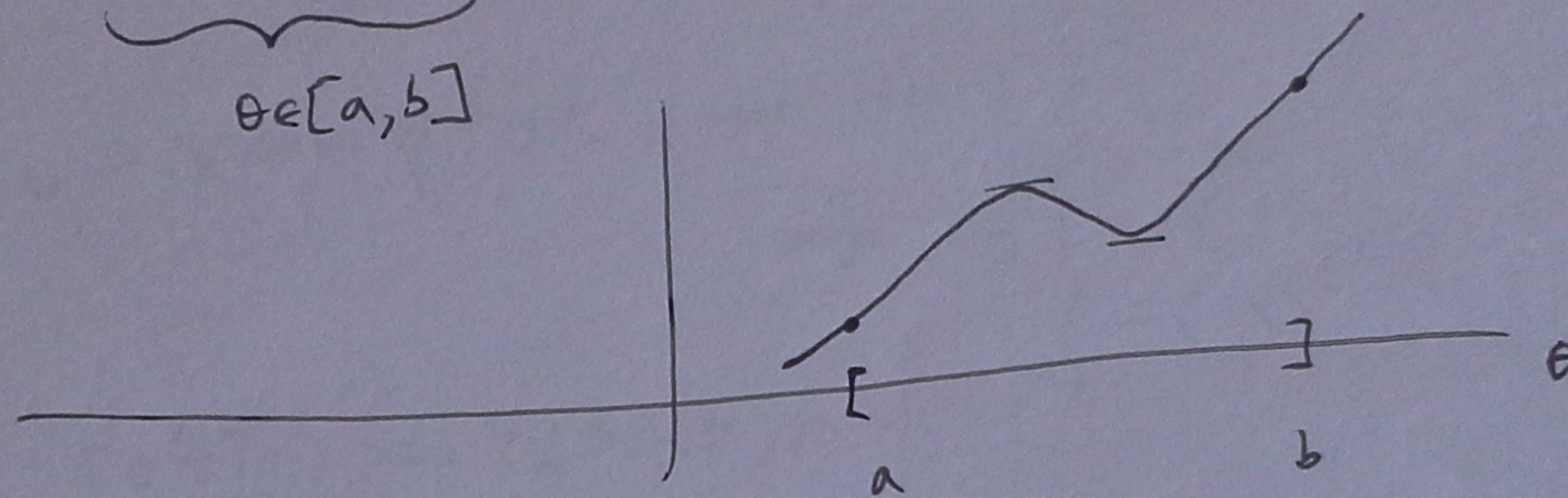
maximize a function $L(\theta)$



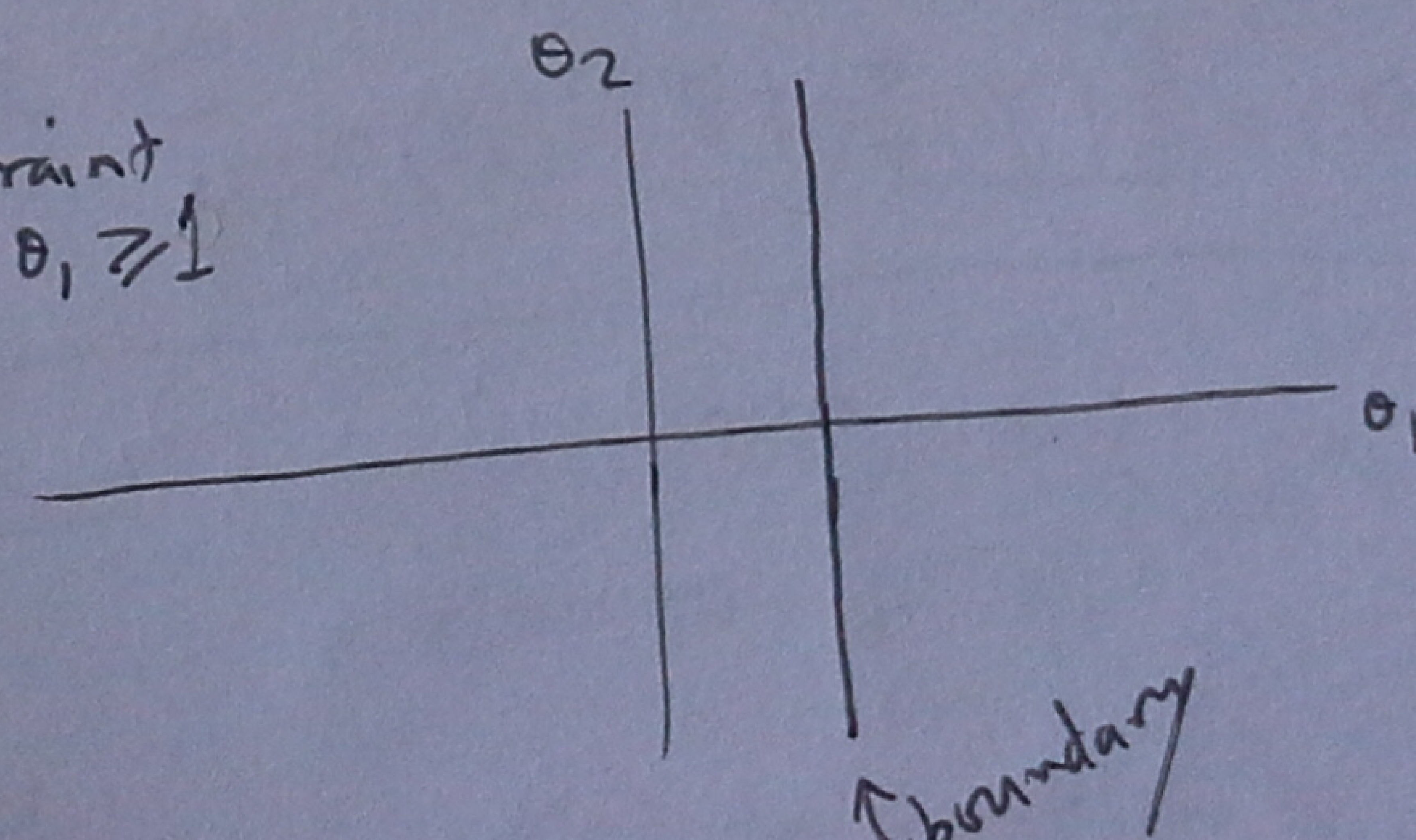
$\theta \in [a,b]$



$$\max_{\theta \in [a,b]} L(\theta) = L(b)$$

Two parameters $\theta = (\theta_1, \theta_2)$

constraint say $\theta_1 \geq 1$



$X_i \sim \text{iid Uniform}(0,\theta)$ Range of $X_i = [0,\theta]$

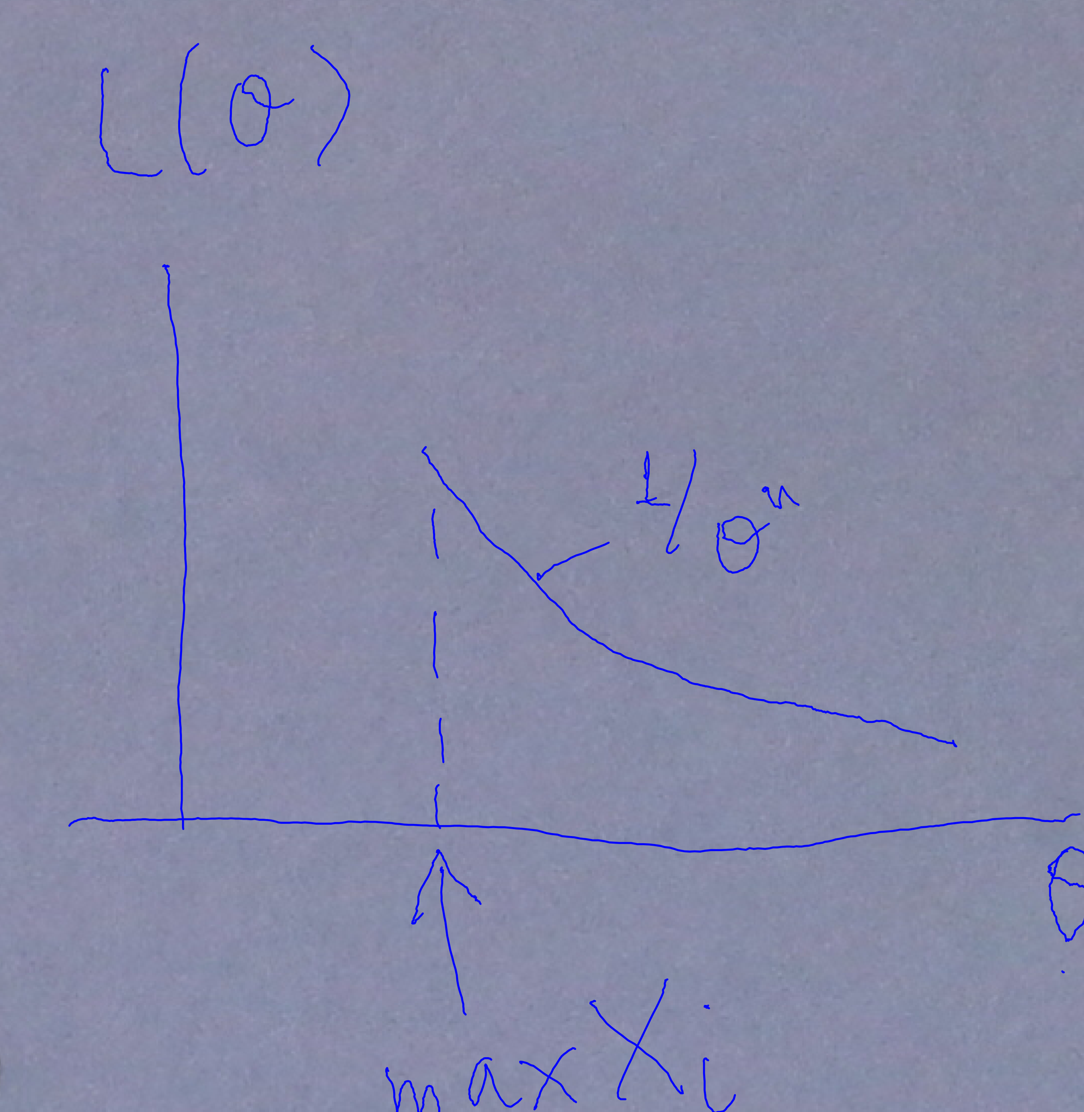
$\uparrow$ depends on θ

$$f_\theta(x) = \begin{cases} 1/\theta & \text{if } 0 < x < \theta \\ 0 & \text{else} \end{cases}$$

$X_i \le \theta \Leftrightarrow \max X_i \le \theta$

$$\boxed{L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} 1_{[0,\theta]}(X_i) = \frac{1}{\theta^n} 1_{[0,\infty]}(\min X_i)\, 1_{[-\infty,\theta]}(\max X_i)} \quad ☆$$

Indicator $\quad 1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases}$ then $f_\theta(x) = \frac{1}{\theta} 1_{[0,\theta]}(x)$



$$\prod_{i=1}^{n} 1_{[0,\theta]}(X_i) = \begin{cases} 1 & \text{if all of the } X_i \text{ are in } [0,\theta] \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \min X_i \geq 0 \ \& \ \max X_i \leq \theta \\ 0 & \text{else} \end{cases}$$

☆ max of $L(\theta)$ is found at the boundary $\theta = \max X_i$

MLE

Two parameters $\theta = (\theta_1, \theta_2)$ $\qquad L(\theta) = \prod_1^n f_\theta(X_i)$

$$\left.\begin{aligned} \frac{\partial L(\theta)}{\partial \theta_1} &= 0 \\ \frac{\partial}{\partial \theta_2} L(\theta) &= 0 \end{aligned}\right\} \Rightarrow \cdots \quad \text{MLE of } \theta$$

$$\boxed{\begin{aligned}\theta_1 &= \mu \\ \theta_2 &= \sigma^2\end{aligned}}$$

ex $X_1, \dots, X_n \sim \text{iid } N(\underbrace{\mu, \sigma^2}_{\theta})$

$$L(\mu, \sigma^2) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{1}{2}\frac{(X_i - \mu)^2}{\sigma^2}\right\}$$

$$= (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_1^n (X_i - \mu)^2\right\}$$

$$\log L(\mu, \sigma^2) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log \sigma^2 - \frac{1}{2}\frac{1}{\sigma^2}\sum_1^n (X_i - \mu)^2$$

$$\left\{\begin{aligned} &\frac{\partial}{\partial \theta_1}\log L = 0 \Rightarrow \frac{1}{\sigma^2}\sum_1^n (X_i - \mu) = 0 \Rightarrow \sum_1^n X_i = \sum_1^n \mu = n\mu \Rightarrow \boxed{\mu = \frac{1}{n}\sum_1^n X_i = \bar{X}} \\ &\qquad\qquad \text{plug-in} \\ &\frac{\partial}{\partial \theta_2}\log L = 0 \Rightarrow -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2}\left[\frac{1}{\sigma^2}\right]^2 \sum_1^n (X_i - \mu)^2 = 0 \Rightarrow \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum (X_i - \bar{X})^2 = 0 \end{aligned}\right.$$

$$\Rightarrow \boxed{\sigma^2 = \frac{1}{n}\sum (X_i - \bar{X})^2}$$

RECAP: Estimation

MOM: based on LLN i.e. that $\frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow EX_i$ when $X_1,\dots,X_n \sim$ iid F with finite $E|X|$

Can estimate EX_i by $\frac{1}{n}\sum X_i$

" " $\underbrace{EX_i^m}_{\boxed{\mu_m}}$ by $\underbrace{\frac{1}{n}\sum X_i^m}_{\boxed{\hat{\mu}_m}}$ (need finite $E|X|^m$, $m>0$)

μ_m (moment of order m) $\qquad$ m^{th} sample moment $\hat{\mu}_m$

ex. $X_1,\dots,X_n \sim$ iid $f_\theta(x)$ ← unknown parameter

Express θ as a function of $\mu_1, \mu_2, \dots$ i.e. $\theta = g(\mu_1, \mu_2, \dots)$

plug-in sample moments

$$\boxed{\hat{\theta}_{MOM} = g(\hat{\mu}_1, \hat{\mu}_2, \dots)}$$

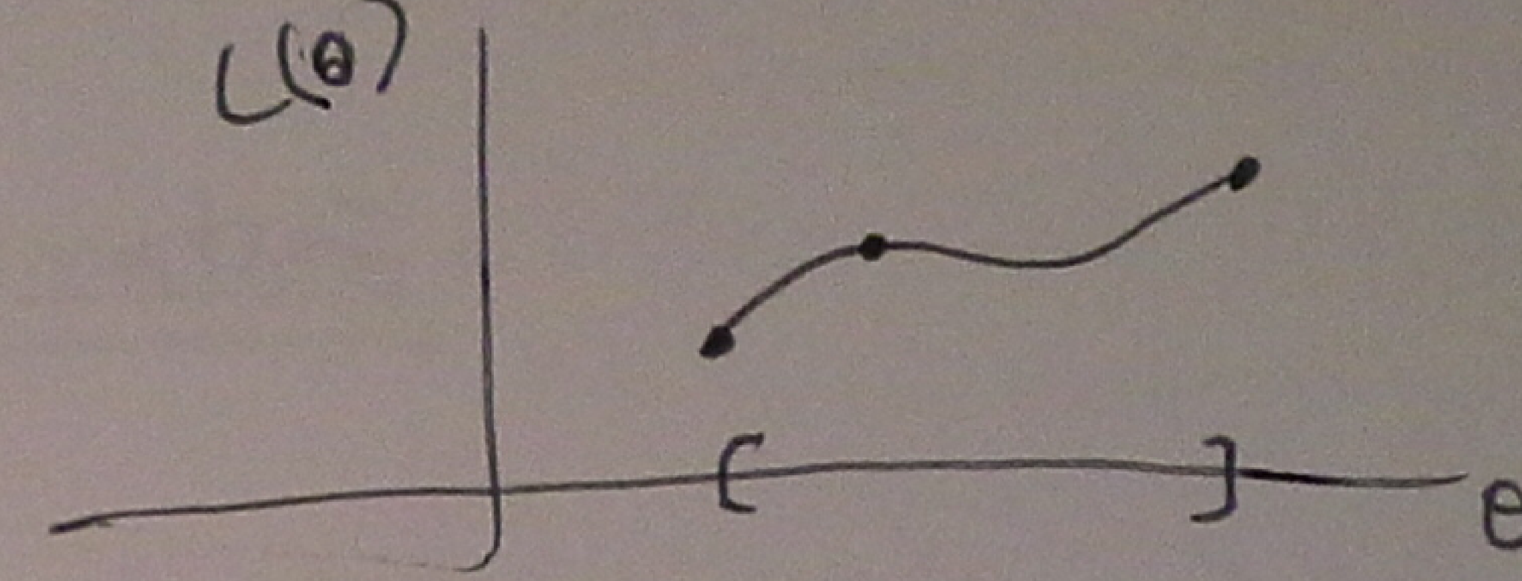
ex. 5.2.5 $f_\theta(x) = \theta x^{\theta-1}$, $0 \le x \le 1$

Compute $\mu_1, \mu_2, \dots$

$$\mu_1 = EX_1 = \int_0^1 \theta x^\theta dx = \frac{\theta}{\theta+1} \Longrightarrow \text{solve for } \theta = \mu_1(\theta+1) \Longrightarrow \theta = \frac{\mu_1}{1-\mu_1} = g(\mu_1)$$

$$\hat{\theta}_{MOM} = g(\underbrace{\tfrac{1}{n}\textstyle\sum X_i}_{\bar{X}}) = \frac{\bar{X}}{1-\bar{X}}$$

MLE $\quad X_1, \dots, X_n \sim \text{iid } f_\theta(x)$



$$L(\theta) = \underbrace{\prod_{i=1}^{n} f_\theta(X_i)}_{\text{joint density evaluated at the data points } X_1, \dots, X_n}$$

MLE: Choose θ that maximizes $L(\theta)$ (or $\log L(\theta)$)

Calculus: Set $\frac{\partial}{\partial \theta} \log L(\theta) = 0$ & solve for $\hat{\theta}_{MLE}$ (but also check boundaries)

e.g. $f_\theta(x) = \begin{cases} g_\theta(x) & \text{if } a(\theta) \le x \le b(\theta) \\ 0 & \text{else} \end{cases} \Rightarrow f_\theta(x) = g_\theta(x)\, 1_{[a(\theta), b(\theta)]}(x)$

$$L(\theta) = \left(\prod_{i=1}^{n} g_\theta(X_i) \right) \left(\underbrace{\prod_{i=1}^{n} 1_{[a(\theta), b(\theta)]}(X_i)}_{= 1_{[a(\theta), \infty)}(\min X_i)\, 1_{(-\infty, b(\theta)]}(\max X_i)} \right)$$

plug-in $L(\theta)$

$a(\theta) \le \min X_i \iff \theta/3 \le \min X_i \iff \boxed{\theta \le 3 \min X_i}$

e.g. $a(\theta) = \theta/3$ & $b(\theta) = \theta$ $\qquad \max X_i \le b(\theta) = \theta \Rightarrow \boxed{\theta \ge \max X_i}$