**Random Variables**

**Discrete**

- **p.m.f.** \( p_x(k) = P(x = k) \)
  - (probability mass function)
  - a) \( 0 \leq p_x(k) \leq 1 \)
  - b) \( \sum p_x(k) = 1 \) for all possible values \( k \)

**Continuous**

- **p.d.f.** \( f_x(x) \)
  - (probability density function)
  - a) \( f_x(x) \geq 0 \)
  - b) \( \int_{-\infty}^{\infty} f_x(x) dx = 1 \)
  - \( P(a \leq x \leq b) = \int_{a}^{b} f_x(x) dx \)
  - \( P(x = a) = 0 \)

**c.d.f.** \( F_x(x) = P(x \leq x) \)

(cumulative distribution function/same definition for discrete or continuous)

- **Properties:**
  1. \( 0 \leq F_x(x) \leq 1 \)
  2. \( F_x(-\infty) = 0, \quad F_x(+\infty) = 1 \)
  3. \( F_x(x) \) is continuous from the right, i.e. \( F_x(x+\varepsilon) \to F_x(x) \) as \( \varepsilon \downarrow 0 \)
  4. \( F_x(x) \) is non-decreasing

**Also:** \( P(a < x \leq b) = F_x(b) - F_x(a) \)

**Discrete**

\( F_x(x) \) is ladder-like with jumps of size \( p_x(k) \) at the possible \( k \)-values

**Continuous**

\( F_x(x) \) is continuous function and \( f_x(x) = F_x'(x) \)

**Graphical Illustration**

- **Discrete Graph:**
  - \( F_x(4.5) \)
  - \( p_x(4.5) \)
  - Size of jump \( i \) \( p_x(i) \)

- **Continuous Graph:**
  - \( f_x(x) \) at \( x = 0.5, 1.5, 2 \)
  - \( F_x(x) \) at \( x = 0.5, 1, 1.5, 2 \)
**JOINT DISTRIBUTIONS**

Definition c.d.f. \( F_{xy}(x,y) = P(X \leq x, Y \leq y) \) and (i.e. \( \uparrow \))

Joint c.d.f. \(<---\) Marginal c.d.f.

\( X \) and \( Y \) are independent \( \iff \) \( F_{xy}(x,y) = F_X(x) \cdot F_Y(y) \) for all \( x, y \).

---

### Discrete R.V.s

<table>
<thead>
<tr>
<th>Joint p.m.f.</th>
<th>( p_{xy}(x,y) = P(X=x, Y=y) )</th>
</tr>
</thead>
</table>

### Continuous R.V.s

|Joint p.d.f.| \( f_{xy}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{xy}(x,y)\) |

**Defining Property of Joint P.D.F.:**

\[ P(\{x\} | \{y\} \in A) = \iint f_{xy}(x,y) \, dx \, dy \]

where \( A \) is a subset of \( \mathbb{R}^2 \).

---

### Marginal (i.e. Individual) P.M.F.

**Marginal** (i.e. individual) P.D.F.

<table>
<thead>
<tr>
<th>P.M.F.</th>
<th>&quot;marginal&quot; (i.e. individual)</th>
</tr>
</thead>
</table>

\[ f_X(x) = \sum_y p_{xy}(x,y) \]

\[ f_Y(y) = \sum_x p_{xy}(x,y) \]

### Conditional P.M.F.

<table>
<thead>
<tr>
<th>Conditional p.m.f.</th>
<th>( p_{xy}(x,y) )</th>
</tr>
</thead>
</table>

\[ P_{xy}(x|y) = \frac{p_{xy}(x,y)}{P_y(y)} \]

\[ P_{yx}(y|x) = \frac{p_{xy}(x,y)}{P_x(x)} \]

\( X \) and \( Y \) are independent \( \iff \) for all \( x, y \)

| Joint p.m.f. | \( p_{xy}(x,y) = p_x(x) \cdot p_y(y) \) |

**Marginal** (i.e. Individual)

\[ f_X(x) \]

\[ f_Y(y) \]

\[ f_{xy}(x,y) = f_X(x) \cdot f_Y(y) \] for all \( x, y \).

| Joint p.d.f. | \( f_{xy}(x,y) = f_X(x) \cdot f_Y(y) \) |

**Marginal** (i.e. Individual)
Properties of Expectation and Variance

Let $X, Y$ be any two r.v.'s and $a, b$ be any two real numbers.

1. $E(aX + bY) = aEX + bEY$
2a. $E(aX + b) = aEX + b$
2b. $\text{Var}(aX + b) = a^2 \text{Var}(X)$

---

If $X, Y$ are independent r.v.'s then we also have:

3a. $E(XY) = (EX)(EY)$
3b. $\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$