ASYMPTOTICS (281B − winter '08)

Def. $\overset{d}{\Rightarrow}$, $\overset{P}{\rightarrow}$, $\overset{a.s.}{\rightarrow}$, $\overset{L^p}{\rightarrow}$

$① \Leftarrow ② \Leftarrow ③$, $④ \Downarrow ②$

← Converse needs uniform integrability (see below) or Dominated convergence

to prov… ② → Chebyche…

To prove ③ → Borel-Cantell… (Complete converg.)

① ⟺ ② when the limit is constant

① ⟺ $F_n(x) \to F(x)$ for all x in a dense set ⟺ $\int g\, dF_n \to \int g\, dF$ $\forall g$ bd. continuous

Pólya: If $F_n \to F$ & F is continuous, then $\|F_n - F\|_\infty \to 0$

Application: If $x_n \to x$ then $F_n(x_n) \to F(x)$.

Continuous map. Th.: If $X_n \overset{d}{\Rightarrow} X$ & g is continuous, then $g(X_n) \overset{d}{\Rightarrow} g(X)$

works for multivariate X too! (or even X = stoch. process)

Convergence of moments

$$\left.\begin{array}{l} F_n \xrightarrow{d} F \\ E|X_n|^{r_0} < M < \infty \end{array}\right\} \Rightarrow \begin{array}{l} E|X_n|^r \to E|X|^r \\ \& \\ E X_n^r \to E X^r \end{array} \quad \forall r \in (0, r_0) \text{ i.e. } 0 < r < r_0$$

$X_n \xrightarrow{P} X \iff \forall$ subsequence $\{n'\}$ has a further subsequence $\{n''\}$ s.t. $X_{n''} \to X$ a.s.

X_n is uniformly integrable (u.i) $\Leftarrow \sup_n E|X_n|^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$

If $|X_n|^p$ is u.i. then ② ⟺ ④

Slutsky: If $\left.\begin{array}{l} Y_n \xrightarrow{P} c_Y \\ Z_n \xrightarrow{P} c_Z \end{array}\right\}$ constants & $X_n \overset{d}{\Rightarrow} X$ then $Y_n X_n + Z_n \overset{d}{\Rightarrow} c_Y X + c_Z$

Dominated convergence $\xrightarrow{P}$ implies $\xrightarrow{L^p}$: If $X_n \xrightarrow{P} X$ & $|X_n| \le |Y|$ a.s. where $E|Y|^r < \infty$ then $X_n \xrightarrow{L^r} X$.

CLT — ch. functions & mgf's (see moment bounds next page)
— Lyapunov & Lindeberg
— Stein's method

Lyapunov CLT Let X_{nk}, $1 \le k \le n$ be row-independent r.v with mean μ_{nk} & variance σ^2_{nk} & $\gamma_{nk} = E|X_{nk} - \mu_{nk}|^3$.
Let: $SD_n = \sqrt{\sum_{k=1}^{n} \sigma^2_{nk}}$, $\gamma_n = \sum_{k=1}^{n} \gamma_{nk}$ & $Z_n = \frac{\sum_{k=1}^{n} (X_{nk} - \mu_{nk})}{SD_n}$

Then, $\gamma_n / SD_n^3 \to 0$ implies $Z_n \xrightarrow{\mathcal{L}} N(0,1)$.

Corollary (Berry–Esseen) Let $X_{n1}, \ldots, X_{nn}$ be iid $F(\mu, \sigma^2)$ & $\gamma = E|X-\mu|^3$

Then $\|F_{Z_n} - \Phi\|_\infty \le \frac{8\gamma}{\sigma^3 \sqrt{n}}$

δ-method If $\tau_n(X_n - \mu) \xrightarrow{\mathcal{L}} X$
& g is differentiable, then $\tau_n[g(X_n) - g(\mu)] \xrightarrow{\mathcal{L}} g'(\mu)X$

Proof: Taylor (what happens if $g'(\mu) = 0$?)
& Slutsky

Fundamental th. of statistics! Taylor: If g has a finite k^{th} derivative at point x, then
$$g(y) = g(x) + \sum_{j=1}^{k} \frac{g^{(j)}(x)}{j!}(y-x)^j + o(|y-x|^k) \quad \text{as } y \to x$$

Edgeworth If F is non-lattice & $\mu_3 = E(X-\mu)^3$ is finite, then
$$P\left(\sqrt{n}\,\frac{\bar{X}-\mu}{\sigma} \le x\right) = \Phi(x) + \frac{\mu_3}{6\sigma^3\sqrt{n}}(1-x^2)\phi(x) + o\left(\frac{1}{\sqrt{n}}\right).$$

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limiting variance fact: Let $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} N(0, v^2)$.
Then $v^2 \le \liminf [n \cdot \operatorname{Var} \hat{\theta}]$

The variance of limit d.f. is $\le$ limiting variance!

δ-method & moments

$\mu_k = E(X-\mu)^k$

Let $h(x) = h(\mu) + h'(\mu)(x-\mu) + \frac{1}{2} h''(\mu)(x-\mu)^2 + R_n$

Let $X_1, \dots, X_n$ be iid(μ, σ^2).

Then $E\,h(\bar{X}) = h(\mu) + h'(\mu)\cancelto{0}{E(\bar{X}-\mu)} + \frac{1}{2} h''(\mu)\,\mathrm{var}\,\bar{X} + O(\frac{1}{n^2})$

to make this precise may need to do a higher order Taylor w. bounds on the highest derivative uniformly

& $\mathrm{var}\,h(\bar{X}) = \dfrac{[h'(\mu)]^2\sigma^2}{n} + \dfrac{h'(\mu)h''(\mu)\mu_3 + \frac{1}{2}[h''(\mu)]^2\sigma^4}{n^2} + O(1/n^3)$

__Note__ $E(\bar{X}-\mu) = 0$, $\mathrm{var}\,\bar{X} = \dfrac{\sigma^2}{n}$

$E(\bar{X}-\mu)^3 = \dfrac{\mu_3}{n^2}$ & $E(\bar{X}-\mu)^4 = \dfrac{\mu_4}{n^3} + \dfrac{3(n-1)\sigma^4}{n^3}$

$$E\left[h(\bar{X}) - E\,h(\bar{X})\right]^3 = \underbrace{\frac{1}{n^2}\left[[h'(\mu)]^3\mu_3 + 3[h'(\mu)]^2\frac{h''(\mu)}{\cancel{2}}\sigma^4\right]}_{} + O(\frac{1}{n^3})$$

__Def:__ h is __normalizing__ if the above is $\ddot{0}$..

__Def.__ h is __variance stabilizing__ if the limiting variance of $h(\bar{X})$ does not depend on unknown parameters.

__e.g.__ Fisher's transf. $h(x) = \frac{1}{2}\log\frac{1+x}{1-x}$ for the correlation in bivariate normal

by Slutsky, $\sqrt{n}(\hat{\rho}-\rho)$ & $\sqrt{n}(\tilde{\rho}-\rho)$ have the same limit d.f.

to verify, write: normalization

$$\hat{\rho} = \frac{\frac{1}{n}\sum(X_i-\bar{X})(Y_i-\bar{Y})}{s_X s_Y} = \underbrace{\boxed{\frac{\frac{1}{n}\sum(X_i-\mu_X)(Y_i-\mu_Y)}{\sigma_X\sigma_Y}}}_{\tilde{\rho}} + \text{negligible}$$

just SAMPLE MEAN!!!

Turn over!