STUDENT'S NAME (please print) :

MATH 281B Final Winter 2008

HAND-IN AT THE T.A.'s (Mike Scullard) MAILBOX ON MONDAY MARCH 17 BY NOON

SOME DEFINITIONS:

An estimator $\hat{\theta}_n$ based on sample $X_1, ..., X_n$ is called **consistent** *for a parameter of interest θ if $\hat{\theta}_n \to \theta$ in probability as $n \to \infty$, i.e., if the estimator achieves it s target in the limit.*

A test is called **consistent** *if it achieves its target in the limit; note that the target of a test is to achieve the desired error rates, i.e., $Prob(TypeI\ error) \to \alpha$ and Power=$1 - Prob(TypeII\ error) \to 1$ as $n \to \infty$.*

Similarly, a confidence interval is called asymptotically **valid** *if its coverage probability tends to the nominal $1 - \alpha$ as $n \to \infty$.*

1. Let $X_1, ..., X_n$ iid $N(\mu, \sigma^2)$—both parameters unknown.

(a) Consider the test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1 (> \theta_0)$ that rejects the null when $\bar{X} > \theta_0 + z(1-\alpha)S/\sqrt{n}$ where $\bar{X}$ is the sample mean, S^2 is the sample variance and $z(1-\alpha)$ is the $1-\alpha$ quantile of the standard normal. Show that this test is *consistent*.

(b) Consider the test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$ using the same critical function, i.e., reject the null when $\bar{X} > \theta_0 + z(1-\alpha)S/\sqrt{n}$. Draw a rough plot of the power function of this test for finite n and superimpose a plot of the power function for a large-sample i.e., as $n \to \infty$.

(c) Consider the confidence interval $\bar{X} \pm z(1-\alpha/2)S/\sqrt{n}$ and show it is an asymptotically *valid* $(1-\alpha)100\%$ confidence interval.

DEFINITIONS (CONTINUED):

Knowing that a test is consistent is very important but it is also nice to know how fast the power tends to one under the alternative. One way to quantify this, is to get some kind of expansion of the type Power=$1 - Prob(TypeIIerror) = 1 + O(1/n^a$ for some $a > 0$. A different (more popular way) is to consider **local alternatives** *(AKA* **contiguous alternatives***).*

To define those, consider testing $H_0 : \theta = \theta_0$ vs. not. A **local alternative** *is given by the hypothesis $K : \theta = \theta_0 + c/n^a$ where $a > 0$ is such that $P(reject\ \ H_0|K)$ tends to a constant in (0,1) as $n \to \infty$. If the test is one-sided, then the sign of the constant c is fixed (positive for alternatives to the right of θ_0, negative else); if the test is two-sided, then c can be either postitive of negative (but not zero).*

2. In the setting of problem 1, find the value of the constant a so that $K : \theta = \theta_0 + c/n^a$ is a **local alternative**. How does $P(reject\ \ H_0|K)$ depend on the constant c?

3. Let $\hat{\theta}_n$ be the MLE of a parameter of interest $\theta \in \Theta$ in a given set-up, and let g be a continuous function. Show that the MLE of $g(\theta)$ is simply $g(\hat{\theta}_n)$. [HINT: partition Θ into sets whose inverse images (under g) lead to the same value for $g(\theta)$; then see in which of those sets $\hat{\theta}_n$ happens to lie.]

4. [Multivariate δ-method]. Let $\hat{\theta}_n$ be a $\sqrt{n}$ consistent, asymptotically normal estimator of a parameter of interest $\theta \in R^d$, i.e, $\sqrt{n}(\hat{\theta}_n - \theta) \Longrightarrow N(0, \Sigma)$ where Σ is a positive definite covariance matrix. Show that if $g : R^d \to R^m$ is a continuously differentiable function then

$$\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \Longrightarrow N(0, \Sigma_g).$$

What is the condition needed to have Σ_g be nontrivial (i.e., not zero) and positive definite? [HINT: recall that when $X = (X_1, ..., X_d)'$ is a d-dimensional r.v. with expected value EX and covariance matrix $V(X) = E(X - EX)(X - EX)'$, and if A, B are non-random matrices/vectors of appropriate dimension, then $E(AX + B) = AEX + B$ and $V(AX + B) = AV(X)A'$.]

5. [Multiparameter LRT test and χ^2 approximation]. Let $X_1, ..., X_n$ be i.i.d. $f_\theta(x)$ where $\theta = (\theta_1, ..., \theta_d)' \in R^d$. Consider the test of $H_0 : \theta_i = 0$ for $i = 1, ..., m(\leq d)$ vs. not. Assume that the unrestricted MLE $\hat{\theta}$ and the restricted (by H_0) MLE $\hat{\theta}_0$ are **jointly** asymptotically multivariate normal with asymptotic covariance matrix Σ. Use problem 4 above to show that $T_n = 2\log[L(\hat{\theta})/L(\hat{\theta}_0)] \Longrightarrow \chi^2_m$ as $n \to \infty$. What should the matrix Σ necessarily be to make the above result happen?

6. [The power family of transformations does not always suffice]. Let $X \sim Binomial(n, p)$ and let $\hat{p} = X/n$. Show that the transformation $g(x) = \arcsin(\sqrt{x})$ stabilizes the variance of $\hat{p}$, while $h(x) = \int_0^x [s(1-s)]^{-\mathbf{1/3}} ds$ is a normalizing transformation. What transformation would you use to derive confidence intervals for p and why? Show the construction of a 95% confidence interval for p using the transformation.

7. [Asumptotic normality of central quantiles]. Let $X_1, ..., X_n$ be i.i.d. from density $f(x)$ and cdf $F(x)$. The α quantile of F is defined as $Q(\alpha, F) = \inf\{x : F(x) \geq \alpha\}$; note that if F is continuous in a neighborhood of $Q(\alpha, F)$, then $Q(\alpha, F) = F^{-1}(\alpha)$ where F^{-1} denotes the inverse function. Let $\hat{F}(x) = (1/n)$ (# of X's $\leq x$) be the empirical d.f.
(a) Find an expression for the α quantile of $\hat{F}$, i.e. $Q(\alpha, \hat{F})$, in terms of an appropriate order statistic; here $\alpha \in (0, 1)$. [HINT: find the closest order statistic to $Q(\alpha, \hat{F})$].
(b) Fix $\alpha \in (0, 1)$ and show that $\sqrt{n}(Q(\alpha, \hat{F}) - Q(\alpha, F)) \Longrightarrow N(0, \tau^2)$ as $n \to \infty$, and identify τ^2. [HINT: this extends the result on asympt. normality of the sample median (obtained with $\alpha = 1/2$) to general central quantiles (i.e., not *extreme* values such as the maximum and the minimum). You may assume for simplicity that $f(x) > 0$ for all x and therefore the inverse function F^{-1} exists,

and thus: $F[Q(\alpha, F)] = \alpha$. The technique to prove this is the same as that for the median: let $G(x - \theta) = F(x)$ where $\theta = Q(\alpha, F)$ so that we now have a location problem $X_1, ..., X_n$ i.i.d. $G(x - \theta)$; let $\hat{\theta} = Q(\alpha, \hat{F})$, and note that

$$P_\theta(\sqrt{n}(\hat{\theta} - \theta) \leq x) = P_0(\sqrt{n}\hat{\theta} \leq x) = P_0(X[\alpha n] \leq x/\sqrt{n}).$$

But the event $X[\alpha n] \leq A$ is equivalent to the event $S_n \leq B$ where $S_n = (\#$ of X's $> A)$ and $B = \lceil (1 - \alpha)n \rceil$. But $P_0(S_n \leq B)$ is a binomial probability...]

8. [The sign test in one and two samples].
(a) Let $X_1, ..., X_n$ be i.i.d. from the *location* family with cdf $F(x - \theta)$ where θ is the median, i.e., $F(0) = 1/2$. Consider the test of $H : \theta = \theta_0$ vs. $K : \theta > \theta_0$ (one-sided) or vs. $K' : \theta \neq \theta_0$ (two-sided). Show that either of these tests can be conducted without distributional assumptions (*nonparametrically*) by letting $Y_i = sign(X_i - \theta)$ and a corresponding (binomial) test on the number of positive signs.
(b) Let $(X_i, Y_i)', i = 1, ..., n$ be i.i.d. observations of the bivariate r.v. $(X, Y)'$ that has distribution $F(x, y)$. Let $\theta_X = EX$ and $\theta_Y = EY$ and consider the test of $H : \theta_X = \theta_Y$ vs. not. Show how this test can be carried out *nonparametrically* by a two-sample sign test. [HINT: Let $D_i = X_i - Y_i$ and use the one-sample sign test.]