MATH 281B Homework

HW1: due in class on Monday Jan 28

1. Show that the exponential family $p_\theta(x) = D(\theta)h(x)\exp(C(\theta)T(x))$ has the monotone likelihood ratio property (MLR) when the function C is monotone.

2. Show that the Binomial (n, θ), Poisson(θ), and Normal (in either parameter, i.e., $N(\theta, a)$ and $N(a, \theta)$ for known a) are members of the above exponential family.

3. Let $X_1, ..., X_n$ be i.i.d. Binomial $(1, \theta)$.
a) Find (in detail) the UMP test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta < \theta_0$ at level $\alpha = 0.05$; do you need a randomized test? Show how the threshold of the critical region is evaluated.
b) Find (in detail) the UMP unbiased test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ at level $\alpha = 0.05$; do you need a randomized test? Show how the thresholds of the critical regions are evaluated. [You may use the slightly suboptimal–but asymptotically optimal–equal tailed test here as it is much easier]

4. Find the P-values of the two tests of ex. 3 (part a and part b) if $\theta_0 = 3/4$ and it was observed that $n^{-1} \sum X_i = 1/4$. (You can leave your answer as a sum).

5. Repeat ex. 3 when $X_1, ..., X_n$ be i.i.d. $N(\theta, 1)$.

6. Repeat ex. 3 when $X_1, ..., X_n$ be i.i.d. $N(0, \theta)$.

7. Use the duality between tests and confidence regions to invert the acceptance regions of the two tests of ex. 3 to get 95% confidence bounds for θ. What kind of confidence bounds are they (lower, upper, interval)? If it is interval, is it symmetric? What kind of optimality do the obtained confidence bounds possess?

8. Repeat the above in the setting of ex. 6, i.e., obtain confidence bounds for the variance of a normal sample.

HW2: due in class on Wed Feb 20–the day of the 281 midterm!

1. Let $X_1, ..., X_n$ be i.i.d. from a distribution that is symmetric around m; so m is the median. Note that the test of $H : m = m_0$ vs. $K : m > m_0$ is equivalent to testing $H^* : p = p_0$ vs. $K^* : p < p_0$ in a related binomial problem.
(a) Identify this binomial problem and find the UMP test there. By the equivalence, this is the test for $H : m = m_0$ vs. $K : m > m_0$.
(b) Invert the test on m to get a UMA confidence bound for the median m.

2. Let $X_1, ..., X_n$ be i.i.d. $N(\mu, \sigma^2)$ where both μ, σ^2 are unknown.
(a) Show that the GLR test is equivalent to the t-test here.
(b) In the above context, find confidence intervals for μ by inverting the GLR test.

3. Let $X_1, ..., X_{n_x}$ be i.i.d. $N(\mu_x, \sigma_x^2)$ and $Y_1, ..., Y_{n_y}$ be i.i.d. $N(\mu_y, \sigma_y^2)$ where the Xs are independent ot the Ys and $\mu_x, \sigma_x^2, \mu_y, \sigma_y^2$ are unknown. Consider the test of $H : \mu_x = \mu_y$ vs. not.
(a) Find the GLR test and show its equivalence to the usual t test.
(b) Assume $\sigma_x^2 = \sigma_y^2$. Find a 95% confidence interval for $\mu_x - \mu_y$ by inverting the GLR test.
(c) Do NOT assume $\sigma_x^2 = \sigma_y^2$. Try to find the GLR test now; can you? What are the difficulties?

4. In the setting of ex. 3 consider the test of $H : \sigma_x^2 \le \sigma_y^2$ vs. $K : \sigma_x^2 > \sigma_y^2$.
(a) Find the GLR test and show it is equivalent to the well-known F test.
(a) Invert the GLR test to find a confidence bound on the ratio σ_x^2/σ_y^2.

5. Let $(X_1, Y_1)', ..., (X_n, Y_n)'$ be bivariate normal with parameters $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ.
(a) Find the GLR test of $H : \mu_X = \mu_X^*, \mu_Y = \mu_Y^*$ vs. $K : \mu_X = \mu_X^+, \mu_Y = \mu_Y^+$ where * and $^+$ denote some fixed values.
(b) Find the GLR test of $H : \mu_X = \mu_X^*, \mu_Y = \mu_Y^*$ vs. not; this test actually reduces to a test on Hotelling's T^2 statistic (a multivariate analog of t^2).

6. In the set-up of ex. 5:
(a) Conduct the GLR test of $H : \rho = 0$ vs. not using the t_{n-2} distribution for $\sqrt{n-2}\hat{\rho}/\sqrt{1-\hat{\rho}^2)}$; what can you say about the power of this test?
(b) Use the large-sample normal distribution of $\hat{\rho}$ to derive confidence intervals for ρ.
(c) Use the large-sample normal distribution of $h(\hat{\rho})$ (where h is Fisher's transformation) to also derive confidence intervals for ρ.
(d) Which of the two intervals (from (b) or (c)) would you prefer and why?? (HINT: to make part (d) simpler you may focus instead on $R = n^{-1} \sum_i (X_i - \mu_X)(Y_i - \mu_Y)/\sigma_X \sigma_Y$ since R and $\hat{\rho}$ have the same asymptotic distribution; note

that R is easier to work with as it is just a sample mean of iid r.v.s!)

7. Let $X_1, ..., X_n$ be iid with mean μ and variance σ^2.
(a) Find a large-sample approximation to the expectation, the variance, and the distribution of $h(\bar{X})$ where h is a smooth (many times differentiable) function.
(b) Apply the above to the case $h(x) = x^2$. Does it matter whether $\mu = 0$ or not? (HINT: yes! Take cases!)