3.2. Let $m_1^{-1}$ and $m_2^{-1}$ be the roots of $\phi(z)$. Writing $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 = (1 - m_1 z)(1 - m_2 z)$, we have $\phi_1 = m_1 + m_2$ and $\phi_2 = -m_1 m_2$ so that the 3 inequalities involving $\phi_1, \phi_2$ can be converted into the following inequalities for $m_1, m_2$:
\[
\begin{align*}
\phi_2 + \phi_1 - 1 &= -m_1 m_2 + m_1 + m_2 - 1 = -(1 - m_1)(1 - m_2) < 0 \\
\phi_2 - \phi_1 - 1 &= m_1 m_2 - m_1 - m_2 - 1 = -(1 + m_1)(1 + m_2) < 0 \\
|\phi_2| &= |m_1 m_2| < 1.
\end{align*}
\]
First assume that $\phi(z)$ is causal, i.e., $|m_1| < 1, |m_2| < 1$. If $m_1$ and $m_2$ are real, then it is immediate that the above inequalities must hold since $(1 \pm m_j) > 0$. On the other hand if $m_1$ and $m_2$ are complex (in which case $m_2 = \overline{m_1}$), then the 3 inequalities are satisfied since
\[
\begin{align*}
-(1 - m_1)(1 - m_2) &= -|1 - m_1|^2 < 0 \\
-(1 + m_1)(1 + m_2) &= -|1 + m_1|^2 < 0 \\
|m_1 m_2| &= |m_1|^2 < 1.
\end{align*}
\]
Conversely, assume $\phi_1$ and $\phi_2$ lie in the region formed by the intersection of the regions determined from the above inequalities. The last inequality, $|m_1 m_2| < 1$, implies that at least one of the $m_j$ (both if $m_1, m_2$ are complex conjugates) must be less than one in absolute value. So assume that $m_1$ and $m_2$ are real and by symmetry take $|m_1| < 1$. Since this implies that $(1 \pm m_1) > 0$, we must have from the first two inequalities that, $(1 \pm m_2) > 0$ or $|m_2| < 1$ as desired.

Now define
\[
\gamma_x(h) = \frac{\sigma^2}{\phi^2 - 1} \phi^{-|h|}.
\]

Now define
\[
\tilde{Z}_t = X_t - \phi^{-1} X_{t-1}.
\]

Then $E\tilde{Z}_t = EX_t - \phi^{-1} EX_{t-1} = 0$ and
\[
\text{Cov}(\tilde{Z}_{t+h}, \tilde{Z}_t) = \text{Cov}(X_{t+h} - \phi^{-1} X_{t+h-1}, X_t - \phi^{-1} X_{t-1})
\]
\[
= \gamma_x(h) - \phi^{-1} \gamma_x(h - 1) - \phi^{-2} \gamma_x(h + 1) + \phi^{-1} \gamma_x(h)
\]
\[
= \begin{cases} 
\frac{\sigma^2}{\phi^2 - 1} (1 - 2\phi^{-2} + \phi^{-2}) = \frac{\sigma^2}{\phi}, & \text{if } h = 0, \\
\frac{\sigma^2}{\phi^2 - 1} (\phi^{-h} - \phi^{-h} - \phi^{-h-2} + \phi^{-2-h}) = 0 & \text{if } h > 0.
\end{cases}
\]

Thus $\{\tilde{Z}_t\} \sim \text{WN}(0, \hat{\sigma}^2)$ with $\hat{\sigma}^2 = \sigma^2/\phi^2$. Note that the causal representation has a smaller white noise variance than the noncausal representation.
3.4. Suppose there exists a stationary solution \( \{X_t\} \) to the equation \( X_t = \phi X_{t-1} + Z_t \) with \( |\phi| = 1 \). Then
\[
X_t = Z_t + \phi Z_{t-1} + \cdots + \phi^{t-1} Z_1 + \phi^t X_0
\]
or
\[
X_t - \phi^t X_0 = Z_t + \phi Z_{t-1} + \cdots + \phi^{t-1} Z_1.
\]
Since \( \{X_t\} \) is assumed to be stationary,
\[
\text{Var}(X_t - \phi^t X_0) = 2\gamma(0) - 2\phi^t \gamma(t) \leq 4\gamma(0).
\]
But the left hand side is equal to
\[
\text{Var}(Z_t + \phi Z_{t-1} + \cdots + \phi^{t-1} Z_1) = (1 + \phi^2 + \cdots + \phi^{2t-2}) \sigma^2 = t\sigma^2
\]
(since \( |\phi| = 1 \)) so that
\[
t\sigma^2 \leq 4\gamma(0)
\]
for all \( t > 0 \). Letting \( t \to \infty \) implies that \( \gamma(0) = \infty \), a contradiction.

3.10. (a) Clearly, \( EX_t = EY_t - .4EY_{t-1} = 0 \) and \( EW_t = 0 \). Also, letting \( \gamma(h) \) be the acf of \( \{Y_t\} \), we have for \( h \geq 0 \)
\[
\gamma_x(h) = \text{Cov}(X_{t+h}, X_t) = \text{Cov}(Y_{t+h} - .4Y_{t+h-1}, Y_t - .4Y_{t-1}) = (1 + (.4)^2)\gamma(h) - .4\gamma(h+1) - .4\gamma(h-1)
\]
and similarly,
\[
\gamma_w(h) = (1 + (2.5)^2)\gamma(h) - 2.5\gamma(h+1) - 2.5\gamma(h-1).
\]
(b) We have
\[
\rho_x(h) = \frac{1.16\gamma(h) - .4(\gamma(h+1) + \gamma(h-1))}{1.16\gamma(0) - .8\gamma(1)} = \frac{.16(7.25\gamma(h) - 2.5(\gamma(h+1) + \gamma(h-1)))}{.16(7.25\gamma(0) - 5\gamma(1))} = \rho_w(h).
\]
(c) By Problem 2.3, the series \( -\sum_{j=1}^{\infty}(.4)^j X_{t+j} \) exists in mean square and
\[
U_t - 2.5U_{t-1} = -\sum_{j=1}^{\infty}(.4)^j X_{t+j} + (.4)^{-1}(.4)X_t + (.4)^{-1}\sum_{j=2}^{\infty}(.4)^{-1}X_{t-1+j} = X_t.
\]

3.13. (a) From (3.3.3) and (3.3.4), \( \psi_j \) satisfies the difference equation
\[
\psi_j - .5\psi_{j-1} + .04\psi_{j-2} = 0, \quad \text{for } j \geq 2,
\]
with boundary conditions,
\[
\psi_0 = 1, \quad \psi_1 = \phi_1 + \theta_1 = .75.
\]
The solution is \( \psi_j = A(.4)^j + B(.1)^j, j \geq 0 \), where \( A \) and \( B \) are found from
\[
A + B = 1, \quad .4A + .1B = .75.
\]
Solving, we obtain \( \psi_j = \frac{13}{9}(.4)^j - \frac{7}{9}(.1)^j. \)
3.14. It is easy to see that process is causal so that \( \gamma(j) \) must satisfy the difference equation

\[
\gamma(j) - \gamma(j-1) + 0.29\gamma(j-2) - 0.02\gamma(j-3) = 0, \quad j \geq 3
\]

with boundary conditions

\[
\gamma(0) - \gamma(1) + 0.29\gamma(2) - 0.02\gamma(3) = 1
\]
\[
\gamma(1) - \gamma(0) + 0.29\gamma(1) - 0.02\gamma(2) = 0
\]
\[
\gamma(2) - \gamma(1) + 0.29\gamma(0) - 0.02\gamma(1) = 0.
\]

The general solution is given by \( \gamma(j) = A(.5)^j + B(.4)^j + C(.1)^j \), where the coefficients are determined from the boundary conditions. Solving these equations, we get \( \gamma(j) = 10.965(.5)^j - 8.267(.4)^j + 0.923(.1)^j \), \( j \geq 0 \).

<table>
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<tr>
<th>( h )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
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<td>( \gamma(h) )</td>
<td>2.790</td>
<td>2.185</td>
<td>1.419</td>
<td>.842</td>
<td>.473</td>
</tr>
</tbody>
</table>

3.15. \( EX_t = 2 + 1.3EX_{t-1} - 4EX_{t-2} + EZ_t + EZ_{t-1} \) so that with \( \mu = EX_t \), we have

\[
\mu = 2 + 1.3\mu - 4\mu
\]

or \( \mu = 20 \). Therefore \( \{Y_t := X_t - 20\} \) is an ARMA(2,1) process which satisfies the equations

\[
Y_t - 1.3Y_{t-1} + 0.4Y_{t-2} = Z_t + Z_{t-1}.
\]

Since \( \theta(z) = 1 + z = 0 \) when \( z = -1 \) the process is not invertible. On the other hand, the process is causal since \( \phi(z) = (1 - .5z)(1 - .8z) = 0 \) when \( z = 2, 1.25 \). The acf \( \gamma(j) \) satisfies the difference equation

\[
\gamma(j) - 1.3\gamma(j-1) + 0.4\gamma(j) = 0, \quad j \geq 2
\]

with boundary conditions

\[
\gamma(0) - 1.3\gamma(1) + 0.4\gamma(2) = \sigma^2(1 + 2.3)
\]
\[
\gamma(1) - 1.3\gamma(0) + 0.4\gamma(1) = \sigma^2.
\]

The general solution to this difference equation is given by \( \gamma(j) = A(.5)^j + B(.8)^j \) which upon solving the boundary conditions

\[
A + B - 1.3(A(.5) + B(.8)) + 0.4(A(.5)^2 + B(.8)^2) = 3.3\sigma^2
\]
\[
A(.5) + B(.8) - 1.3(A + B) + 0.4(A(.5) + B(.8)) = \sigma^2.
\]

for \( A \) and \( B \) yields \( \gamma(j) = \sigma^2(-16.666(.5)^j + 50(.8)^j) \).
3.22. The ACVF of \( \{X_t\} \) is

\[
\gamma(0) = \sigma^2(1 + \theta^2)
\]

\[
\gamma(1) = \sigma^2\theta
\]

\[
\gamma(k) = 0, \quad k > 1.
\]

By the projection theorem, the best predictor in \( \mathbb{S}_p\{X_1, \ldots, X_n\} \) of \( X_{n+1} \) is

\[
\hat{X}_{n+1} = \sum_{j=1}^{n} \phi_j X_{n+1-j},
\]

where \( \phi := (\phi_1, \ldots, \phi_n)' \) satisfies

\[
\Gamma_n \phi = \gamma_n,
\]

i.e.

\[
\begin{bmatrix}
1 + \theta^2 & -\theta & 0 & \cdots & 0 \\
-\theta & 1 + \theta^2 & -\theta & \cdots & 0 \\
0 & -\theta & 1 + \theta^2 & -\theta & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & -\theta \\
0 & 0 & \cdots & 0 & 1 + \theta^2 \\
0 & 0 & \cdots & 0 & 1 + \theta^2 \\
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_3 \\
\vdots \\
\phi_n \\
\end{bmatrix}
= \begin{bmatrix}
-\theta \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

This equation is equivalent to the \( n \) equations specified.

3.23. From Definition 3.4.2 the value of the PACF at lag \( n \) is \( \phi_n \) where \( \{\phi_j, j = 1, \ldots, n\} \) satisfies the difference equations and boundary conditions derived in Problem 3.22.

The general solution of the difference equations is

\[
\phi_j = A \lambda_1^j + B \lambda_2^j, \quad j = 1, \ldots, n,
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the roots of

\[
\lambda^2 - (\theta + \theta^{-1})\lambda + 1 = 0,
\]

i.e. \( \lambda_1 = \theta \) and \( \lambda_2 = \theta^{-1} \). The constants \( A \) and \( B \) are determined by the boundary conditions,

\[
(1 + \theta^2)\phi_n - \theta\phi_{n-1} = 0,
\]

\[
(1 + \theta^2)\phi_1 - \theta\phi_2 = -\theta,
\]

which reduce to

\[
A\theta^n + B\theta^{-n} = 0,
\]

\[
A + B = -1.
\]

Solving, we obtain \( A = -(1 - \theta^{2n+2})^{-1} \) and \( B = \theta^{2n+2} (1 - \theta^{2n+2})^{-1} \). Hence the PACF at lag \( n \) is

\[
\alpha(n) = \phi_n = -\theta^n \frac{1 - \theta^2}{1 - \theta^{2n+2}}.
\]