Supplement to "Skip-sampling: subsampling in the frequency domain"

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Appendix A: Proofs of Main Results

Throughout, let $\stackrel{\mathcal{L}}{\Longrightarrow}$ denote convergence in law and $\stackrel{P}{\longrightarrow}$ denote convergence in probability.

Proof of Theorem 3.1. Let x be a point of continuity of J. By the first argument given in the proof of Theorem 2.2.1 of Politis et al. (1999), $U_{b,T}(x)$ and $L_{b,T}(x)$ are asymptotically close. So to prove Theorem 3.1, it suffices to show that $U_{b,T}(x) \stackrel{P}{\longrightarrow} J(x)$.

By Assumption (A*) and the Cesaro sums lemma,

$$EU_{b,T}(x) = q^{-1} \sum_{j=1}^{q} P[a_b(\widehat{\theta}_b^{(j)} - \theta) \leq x] = J_b(x) + o(1).$$

Furthermore, $J_b(x) \to J(x)$ by Assumption (A) and eq. (5). Now

$$\mathrm{Var}(U_{b,T}(x)) = q^{-2} \sum_{j=1}^{q} \sum_{k=1}^{q} \mathrm{Cov}(g_1(\widehat{\theta}_b^{(j)}), g_1(\widehat{\theta}_b^{(k)})),$$

where $g_1(\widehat{\theta}_b^{(j)}) = \mathbf{1}\{a_b(\widehat{\theta}_b^{(j)} - \theta) \leq x\}$. The double summation breaks into a single summation (taking $j = k$) and the remaining terms (where $j \neq k$). For this latter summand, we have

$$q^{-2} \sum_{j \neq k=1}^{q} \mathrm{Cov}(g_1(\widehat{\theta}_b^{(j)}), g_1(\widehat{\theta}_b^{(k)})),$$

which can be expressed as a single sum of $q^q - q$ terms, each of which tends to zero as $T \to \infty$ (by Assumption (A*)), which also implies that $q \to \infty$ by eq. (5). Therefore this is a Cesaro sum, and is $o(1)$; hence,

$$\mathrm{Var}(U_{b,T}(x)) = q^{-2} \sum_{j=1}^{q} \mathrm{Cov}(g_1(\widehat{\theta}_b^{(j)}), g_1(\widehat{\theta}_b^{(j)})) + o(1).$$

Since g_1 is an indicator, it follows that $|\mathrm{Cov}(g_1(\widehat{\theta}_b^{(j)}), g_1(\widehat{\theta}_b^{(j)}))| \leq 1$. Hence, $\mathrm{Var}(U_{b,T}(x)) = o(1)$, and the desired result follows by Chebyshev's inequality. $\square$

Proof of Corollary 3.1. The oracle quantity $\widetilde{v}_b = \frac{a_b^2}{q}\sum_{j=1}^q(\widehat{\theta}_b^{(j)} - \theta)^2$ is realted to $\widehat{v}_b$ via

$$\widehat{v}_b = \frac{a_b^2}{q}\sum_{j=1}^q(\widehat{\theta}_b^{(j)} - \theta + \theta - \bar{\widehat{\theta}}_b)^2 = \widetilde{v}_b - a_b^2(\bar{\widehat{\theta}}_b - \theta)^2. \tag{A.1}$$

By assumption, $E\bar{\widehat{\theta}}_b = \theta + o(a_b^{-1})$, and

$$\begin{aligned} q^2\mathrm{Var}[\bar{\widehat{\theta}}_b] &= \sum_{i=1}^q\sum_{j=1}^q \mathrm{Cov}[\widehat{\theta}_b^{(i)}, \widehat{\theta}_b^{(j)}] \\ &= \sum_{i=1}^q \mathrm{Var}[\widehat{\theta}_b^{(i)}] + \sum_{i=1}^q\sum_{j\neq i}\mathrm{Cov}[\widehat{\theta}_b^{(i)}, \widehat{\theta}_b^{(j)}] \\ &= qa_b^{-2}(v + o(1)) + O(q^2/T), \end{aligned}$$

so that

$$a_b^2\, E(\bar{\widehat{\theta}}_b - \theta)^2 = \frac{v + o(1)}{q} + O(a_b^2/T),$$

which tends to zero by eq. (7). Hence, eq. (A.1) implies that $\widehat{v}_b - \widetilde{v}_b \xrightarrow{P} 0$.

Next, we show $\widetilde{v}_b \xrightarrow{P} v$: assumption (8) implies

$$E\widetilde{v}_b = \frac{a_b^2}{q}\sum_{j=1}^q\left(\mathrm{Var}[\widehat{\theta}_b^{(j)}] + \mathrm{Bias}^2[\widehat{\theta}_b^{(j)}]\right) = v + o(1),$$

i.e., $\mathrm{Bias}[\widetilde{v}_b] \to 0$. To show $\mathrm{Var}[\widetilde{v}_b] \to 0$, write

$$\widetilde{v}_b - v = \frac{a_b^2}{q}\sum_{j=1}^q(\widehat{\theta}_b^{(j)} - \theta)^2 - v = \frac{1}{q}\sum_{j=1}^q s_j,$$

letting $s_j = a_b^2(\widehat{\theta}_b^{(j)} - \theta)^2 - v$. Finally,

$$E(\widetilde{v}_b - v)^2 = \frac{1}{q^2}\sum_{i=1}^q\sum_{j=1}^q E(s_is_j) = \frac{1}{q^2}\sum_{i=1}^q Es_i^2 + \frac{1}{q^2}\sum_{1\leq i\neq j\leq q} E(s_is_j) = O(1/q) + o(1)$$

by (9). By Chebyshev's inequality, it follows that $\widetilde{v}_b \xrightarrow{P} v$, and therefore $\widehat{v}_b \xrightarrow{P} v$ as well. $\square$

Proof of Theorem 3.2. The proof is similar to that of Theorem 3.1, and we focus on the variance calculation of $U^{\sharp}_{b,T}(x)$. Using the identity

$$\sum_{r=0}^{n} \binom{n}{r} r a^{n-r} b^{r} = bn(a+b)^{n-1},$$

we obtain

$$\begin{aligned}
\mathrm{Var}(U^{\sharp}_{b,T}(x)) &= S^{-2} \sum_{j=1}^{S} \sum_{k=1}^{S} \mathrm{Cov}(g_1(\widehat{\theta}_b^{(\sharp j)}), g_1(\widehat{\theta}_b^{(\sharp k)})) \\
&= S^{-2} \sum_{r=0}^{b} \binom{b}{r} (q-1)^{b-r} S\, O(r/b) \\
&= O(1)\, S^{-1} b^{-1} \sum_{r=0}^{b} \binom{b}{r} (q-1)^{b-r} r \\
&= O(1)\, S^{-1} b^{-1}\, b q^{b-1} = O(1/q),
\end{aligned}$$

which tends to zero. Now the result follows from Chebyshev's inequality. □

Proof of Theorem 4.1. Assumption (Bk) is used implicitly in the cumulant calculations below. Let $d_T(\lambda) = \sum_{t=1}^{T} X_t e^{-it\lambda}$, so that for $\lambda \neq 0$ we have $I_T(\lambda) = T^{-1}|d_T(\lambda)|^2$. When evaluated at a non-zero Fourier frequency λ_ℓ, $d_T(\lambda_\ell)$ gives the same value when computed from $X_t - \mu$ instead of X_t, because $\sum_{t=1}^{T} e^{-it\lambda_\ell} = 0$. So without loss of generality we can assume that $\mu = 0$ in our analysis. Furthermore, because $E[d_T(\lambda) d_T(-\lambda)] = \mathrm{cum}(d_T(\lambda), d_T(-\lambda))$ when $\mu = 0$, we can apply Theorem 4.3.2 of Brillinger (1981) to obtain

$$\begin{aligned}
E[\widehat{\theta}_b^{(j)}] &= \frac{1}{bT} \sum_{\ell=1}^{[b/2]} g^{\star}(2\pi\ell/b)\, E[d_T(2\pi\ell/b + 2\pi\tilde{j}/T) d_T(-2\pi\ell/b - 2\pi\tilde{j}/T)] \\
&= \frac{1}{bT} \sum_{\ell=1}^{[b/2]} g^{\star}(2\pi\ell/b)\, \big(T f(2\pi\ell/b + 2\pi\tilde{j}/T) + O(1)\big) \\
&= O(T^{-1}) + \frac{1}{b} \sum_{\ell=1}^{[b/2]} g^{\star}(2\pi\ell/b)\, f(2\pi\ell/b + 2\pi\tilde{j}/T),
\end{aligned}$$

using the boundedness of g. The case $k = 2$ of Assumption (B) implies that $\partial_\lambda f(\lambda)$ is bounded in λ, and hence by a Taylor series expansion $f(2\pi\ell/b + 2\pi\tilde{j}/T) = f(2\pi\ell/b) + O(b^{-1})$ uniformly in ℓ, since $|\tilde{j}|/T \leq q/(2T) = 1/(2b)$. Thus, $E[\widehat{\theta}_b^{(j)}] = b^{-1} \sum_{\ell=1}^{[b/2]} g^{\star}(2\pi\ell/b)\, f(2\pi\ell/b) + O(b^{-1}) + O(T^{-1})$. Finally,

$$b^{-1} \sum_{\ell=1}^{[b/2]} g^{\star}(2\pi\ell/b)\, f(2\pi\ell/b) = b^{-1} \sum_{|\ell|=1}^{[b/2]} g(2\pi\ell/b)\, f(2\pi\ell/b)$$

by the evenness of f, and this last expression is the Riemann sum on a mesh of size b^{-1} of $\langle gf \rangle = \theta$. This proves the first assertion.

Covariance of skip-sample statistics. Next, consider for any $i, j = 1, \ldots, q$ the covariance:

$$\begin{aligned}\mathrm{Cov}\left(\widehat{\theta}_b^{(i)}, \widehat{\theta}_b^{(j)}\right) = \frac{1}{b^2 T^2} \sum_{\ell_1, \ell_2 = 1}^{[b/2]} & g^\star(2\pi\ell_1/b)\, g^\star(2\pi\ell_2/b) \\ & \cdot \mathrm{cum}\left\{ d_T(2\pi\ell_1/b + 2\pi\tilde{i}/T) d_T(-2\pi\ell_1/b - 2\pi\tilde{i}/T), \right. \\ & \quad \left. d_T(2\pi\ell_2/b + 2\pi\tilde{j}/T) d_T(-2\pi\ell_2/b - 2\pi\tilde{j}/T) \right\}.\end{aligned}$$

To compute the cumulant, we use Theorems 2.3.2 and 4.3.2 of Brillinger (1981); we write the four frequencies in a 2×2 table:

$$\mathcal{T}_{2,2} = \left[\begin{array}{cc} \ell_1/b + \tilde{i}/T & -\ell_1/b - \tilde{i}/T \\ \ell_2/b + \tilde{j}/T & -\ell_2/b - \tilde{j}/T \end{array} \right].$$

Here, the entries of $\mathcal{T}_{2,2}$ correspond to the frequencies in the cumulant expression, but divided by 2π. Following the arguments used in the proofs of Propositions A.1 and A.2 in McElroy and Roy (2022), we seek indecomposable partitions of the table, and can restrict our attention to either (i) two 2-sets or (ii) a 4-set. (A p-set is a set in the partition with p elements; we need not consider 1-sets, because the cumulant of a single DFT is the expectation, which is zero.) Letting $\sharp$ and $\flat$ denote membership in a particular 2-set, the only indecomposable partitions in case (i) are

$$\left[\begin{array}{cc} \sharp & \flat \\ \sharp & \flat \end{array} \right], \quad \left[\begin{array}{cc} \sharp & \flat \\ \flat & \sharp \end{array} \right].$$

If the sum of the frequencies in a particular 2-set equal zero, then we can apply Theorem 4.3.2 of Brillinger (1981) to replace those two terms of the cumulant by T times f evaluated at either of the two frequencies. Before proceeding further, we state a simple lemma; here, we denote $q\mathbb{Z}$ to be the set of all integers that are divisible by q.

Lemma A.1 *If $i.j = 1, \ldots, q$, then $\tilde{j} - \tilde{i} \in q\mathbb{Z}$ implies $i = j$.*

Proof of Lemma A.1. First, $\tilde{j} - \tilde{i} = j - i$. Since $\tilde{j} - \tilde{i} = q\ell$ for some $\ell \in \mathbb{Z}$, $j = i + q\ell$ and $i = j - q\ell$. Since $j \leq q$, the first expression implies that $\ell = 0, -1$. Since $i \leq q$, the second expression implies that $\ell = 0, 1$. Hence $\ell = 0$ and $i = j$. $\Box$

Returning to the theorem's proof, we see that for the first partition of case (i) the frequencies sum to zero (in both 2-sets) if and only if $q(\ell_1 + \ell_2) + (\tilde{i} + \tilde{j}) = 0$, which can never happen (since $\ell_1, \ell_2 \geq 1$). For the second partition of case (i), the condition is that $q(\ell_1 - \ell_2) + (\tilde{i} - \tilde{j}) = 0$, which by Lemma A.1 is impossible unless $i = j$, in which case we must have $\ell_1 = \ell_2$. Now $i = j$ pertains to the variance calculation; in that case, ignoring all lower order terms, we have the term

$$\frac{1}{b^2 T^2} \sum_{\ell_1 = 1}^{[b/2]} g^\star(2\pi\ell_1/b)^2 \left\{ T f(2\pi\ell_1/b + 2\pi\tilde{i}/T) + O(1) \right\}^2$$

contributing to the variance of $\widehat{\theta}_b^{(i)}$. The double sum has collapsed to a single sum because of the constraint that $\ell_1 = \ell_2$.

However, we must sum over all indecomposable partitions, and the 4-set of case (ii) must still be accounted for. This 4-set consists of all four frequencies, and the sum is of course zero, no matter whether $\tilde{i} = \tilde{j}$ or not. Again by Theorem 4.3.2 of Brillinger (1981), we can replace the entire cumulant expression by T times F evaluated at the first three frequencies, viz. entries $(1,1)$, $(1,2)$, and $(2,1)$ of $\mathcal{T}_{2,2}$ (multiplied by 2π). Combining over all indecomposable partitions, we obtain

$$\mathrm{Cov}\left(\widehat{\theta}_b^{(i)}, \widehat{\theta}_b^{(j)}\right) = \frac{1}{b^2} \sum_{\ell_1=1}^{[b/2]} g^\star(2\pi\ell_1/b)^2 \, f(2\pi\ell_1/b + 2\pi\tilde{i}/T)^2 \, \mathbf{1}\{i=j\} + O(b^{-1}T^{-1})$$
$$+ \frac{1}{b^2 T} \sum_{\ell_1,\ell_2=1}^{[b/2]} g^\star(2\pi\ell_1/b) \, g^\star(2\pi\ell_2/b) \cdot F(2\pi\ell_1/b + 2\pi\tilde{i}/T, -2\pi\ell_1/b - 2\pi\tilde{i}/T, 2\pi\ell_2/b + 2\pi\tilde{j}/T) + O(T^{-2}).$$

Ignoring the error terms, the first summand (which only occurs if $i = j$) is $O(b^{-1})$, using properties of the Riemann sum and the smoothness of f. Similarly, the second summand is $O(T^{-1})$. (Assumption B establishes the summability of higher order autocumulants, and correspondingly the smoothness of polyspectra.) Both summands can be rewritten as integrals at the cost of incurring $O(b^{-1})$ error multiplying their overall order of decay; this finally yields

$$\mathrm{Cov}\left(\widehat{\theta}_b^{(i)}, \widehat{\theta}_b^{(j)}\right) = \frac{1}{2\pi b} \int_0^\pi g^\star(\lambda)^2 \, f(\lambda)^2 d\lambda \, \mathbf{1}\{i=j\} + O(b^{-2}) + O(b^{-1}T^{-1})$$
$$+ \frac{1}{4\pi^2 T} \int_0^\pi \int_0^\pi g^\star(\lambda) \, g^\star(\omega) \, F(\lambda, -\lambda, \omega) \, d\lambda d\omega + O(b^{-1}T^{-1}) + O(T^{-2}).$$

Using the symmetry in f and the definition of $g^\star$, we obtain

$$\frac{1}{2\pi} \int_0^\pi (g(\lambda) + g(-\lambda))^2 \, f(\lambda)^2 d\lambda = \langle g g^\star f^2 \rangle$$
$$\frac{1}{4\pi^2} \int_0^\pi \int_0^\pi g^\star(\lambda) \, g^\star(\omega) \, F(\lambda, -\lambda, \omega) \, d\lambda d\omega = \langle\langle g g F \rangle\rangle,$$

establishing the variance result.

Covariance of skip-sample statistic with a squared skip-sample statistic. Next, we turn to the second covariance calculation, proceeding with more brevity: for any $i, j = 1, \ldots, q$,

$$\mathrm{Cov}\left({\widehat{\theta}_b^{(i)}}^2, \widehat{\theta}_b^{(j)}\right) = \frac{1}{b^3 T^3} \sum_{\ell_1,k_1,\ell_2=1}^{[b/2]} g^\star(2\pi\ell_1/b) \, g^\star(2\pi k_1/b) \, g^\star(2\pi\ell_2/b)$$
$$\cdot \, \mathrm{cum}\left\{ d_T(2\pi\ell_1/b + 2\pi\tilde{i}/T) d_T(-2\pi\ell_1/b - 2\pi\tilde{i}/T) \right.$$
$$\cdot \, d_T(2\pi k_1/b + 2\pi\tilde{i}/T) d_T(-2\pi k_1/b - 2\pi\tilde{i}/T),$$
$$\left. d_T(2\pi\ell_2/b + 2\pi\tilde{j}/T) d_T(-2\pi\ell_2/b - 2\pi\tilde{j}/T) \right\}.$$

Correspondingly, we consider the following table:

$$\mathcal{T}_{4,2} = \left[\begin{array}{cccc} \ell_1/b + \tilde{i}/T & -\ell_1/b - \tilde{i}/T & k_1/b + \tilde{i}/T & -k_1/b - \tilde{i}/T \\ \ell_2/b + \tilde{j}/T & -\ell_2/b - \tilde{j}/T & & \end{array} \right].$$

In the indecomposable partitions, each set in a partition contributes $O(T)$. We focus upon the following cases: (i) three 2-sets (for a total order of T^3), (ii) two 3-sets (for a total order of T^2), and (iii) a 2-set and a 4-set (for a total order of T^2). There are three summations running from 1 to $[b/2]$, each of which contributes $O(b)$ in the final analysis; if a double sum collapses due to a constraint (e.g., two indices must be equal), then there will be one less contribution of $O(b)$ to the total order. Given the $b^3 T^3$ in the denominator, the overall order in b will be $O(b^{-r})$, where $r = 0, 1, 2$ is the number of sums that collapse, and the order in T will be $O(1)$ in case (i) and $O(T^{-1})$ for cases (ii) and (iii).

Because i corresponds to the skip-sample that is squared, whereas the jth skip-sampled is not squared, i and j do not figure symmetrically into the analysis. Beginning with case (i), there are 12 distinct partitions, and for 8 of these the three constaints (that the sum of frequencies belonging to each of three 2-sets equals zero) are never satisfied. For the remaining 4 partitions, we describe the results as follows: let $\sharp$, $\flat$, and $\natural$ denote membership in a 2-set. Then these 4 relevant partitions are

$$\left[\begin{array}{cccc} \sharp & \sharp & \flat & \natural \\ \natural & \flat & & \end{array} \right] \qquad \left[\begin{array}{cccc} \sharp & \flat & \natural & \sharp \\ \flat & \natural & & \end{array} \right]$$
$$\left[\begin{array}{cccc} \sharp & \flat & \flat & \natural \\ \natural & \sharp & & \end{array} \right] \qquad \left[\begin{array}{cccc} \sharp & \flat & \natural & \natural \\ \flat & \sharp & & \end{array} \right]$$

For the upper left partition, one of the conditions (corresponding to $\sharp$) is always satisfied, and the other two conditions are equivalent, viz. $q(k_1 - \ell_2) = \tilde{j} - \tilde{i}$, which holds if and only if $k_1 = \ell_2$ and $i = j$ by Lemma A.1. In such a case, the overall order will be $O(b^{-1})$. The lower right partition has an identical analysis.

For the upper right partition, the three conditions amount to $\ell_1 = k_1$, $q(\ell_2 - \ell_1) = \tilde{i} - \tilde{j}$, and $q(k_1 - \ell_2) = \tilde{j} - \tilde{i}$. So if $i \neq j$ the overall order will be $O(b^{-1} T^{-2})$, which is small enough (compared to other terms) that it can be ignored. But if $i = j$, the three equations amount to two constraints, i.e., $\ell_1 = k_1 = \ell_2$, and the overall order is $O(b^{-2})$. The lower left partition has an identical analysis.

Now considering case (ii), there are 10 distinct partitions, for 7 of which the constraints are never satisfied. The remaining 3 partitions are described below, with $\sharp$ and $\flat$ denoting membership in a 3-set:

$$\left[\begin{array}{cccc} \sharp & \flat & \sharp & \flat \\ \flat & \sharp & & \end{array} \right], \quad \left[\begin{array}{cccc} \sharp & \flat & \flat & \sharp \\ \sharp & \flat & & \end{array} \right], \quad \left[\begin{array}{cccc} \sharp & \flat & \flat & \sharp \\ \flat & \sharp & & \end{array} \right]$$

For the left partition, the two conditions are equivalent to each other, and are $q(\ell_1 - \ell_2 + k_1) = \tilde{j} - 2\tilde{i}$. If $i = j$, then the only solution is with $\tilde{i} = 0$ and $\ell_1 - \ell_2 + k_1 = 0$, yielding $O(b^{-1} T^{-1})$. But if $i \neq j$, then solutions could be $\ell_1 - \ell_2 + k_1$ equal to -1, 0, or 1, depending on whether $\tilde{j} - 2\tilde{i}$ equals $-q$, 0, or q. In any of these situations, the overall order is $O(b^{-1} T^{-1})$. (But if $\tilde{j} - 2\tilde{i}$ is not equal to $-q$, 0, or q, then the three frequencies do not sum to zero.)

For the middle partition, the two conditions are again equivalent, reducing to $q(\ell_1 - k_1 + \ell_2) = -\tilde{j}$, which holds if and only if $\tilde{j} = 0$ and $\ell_1 - k_1 + \ell_2 = 0$. This is a single collapse, and yield $O(b^{-1}T^{-1})$, independent of the value of $\tilde{i}$ (but only if $\tilde{j} = 0$). The analysis of the right partition is similar.

Finally for case (iii), there are 14 distinct partitions, of which 6 can be ignored because the frequency constrainsts are never satisfied. For the remaining 8 partitions, we describe the results using $\sharp$ for the 2-set and $\flat$ for the 4-set:

$$\begin{bmatrix} \sharp & \sharp & \flat & \flat \\ \flat & \flat & & \end{bmatrix} \qquad \begin{bmatrix} \sharp & \flat & \flat & \sharp \\ \flat & \flat & & \end{bmatrix}$$
$$\begin{bmatrix} \sharp & \flat & \flat & \flat \\ \flat & \sharp & & \end{bmatrix} \qquad \begin{bmatrix} \flat & \sharp & \sharp & \flat \\ \flat & \flat & & \end{bmatrix}$$
$$\begin{bmatrix} \flat & \sharp & \flat & \flat \\ \sharp & \flat & & \end{bmatrix} \qquad \begin{bmatrix} \flat & \flat & \sharp & \sharp \\ \flat & \flat & & \end{bmatrix}$$
$$\begin{bmatrix} \flat & \flat & \flat & \sharp \\ \sharp & \flat & & \end{bmatrix} \qquad \begin{bmatrix} \flat & \flat & \flat & \sharp \\ \flat & \sharp & & \end{bmatrix}$$

Moving from left to right, starting at the top row, the first partition has all frequencies sum to zero in both the 2-set and the 4-set without any constraint, and hence the overall order is $O(T^{-1})$. The sixth partition has the same analysis.

For the second partition, the constraint from the 2-set reduces to $\ell_1 = k_1$, and the 4-set constraint is equivalent. Thus the order is $O(b^{-1}T^{-1})$. The fourth partition has the same analysis.

The third partition has the 2-set constraint $q(\ell_1 - \ell_2) = \tilde{j} - \tilde{i}$, which by Lemma A.1 holds if and only if $i = j$ and $\ell_1 = \ell_2$. The 4-set constraint is equivalent to this, so the overall order is $O(b^{-1}T^{-1})$ only when $i = j$. The fifth, seventh, and eighth partitions have similar analyses.

Now we sum over all the contributing indecomposable partitions occurring in cases (i), (ii), and (iii), obtaining terms of order $O(b^{-1}) + O(b^{-2}) + O(b^{-1}T^{-1}) + O(T^{-1})$ when $i = j$, and $O(b^{-1}T^{-1}) + O(T^{-1}) + O(b^{-1}T^{-2})$ otherwise. Focusing on the latter situation, where $i \neq j$ (the case of $i = j$ is not of especial interest to our later applications, and hence is not further described in the theorem's statement), the highest order term is $O(T^{-1})$, which occurs for the first and sixth partitions of case (iii); this proves the theorem's assertion.

Covariance of squared skip-sample statistics. Finally, we turn to the third covariance calculation; the method of proof is similar to the prior cases, but much more tedious due to the increased number of indecomposable partitions.

We provide a summary, omitting the details. For any $i, j = 1, \ldots, q$,

$$
\begin{aligned}
\mathrm{Cov}\left(\widehat{\theta}_b^{(i)\,2}, \widehat{\theta}_b^{(j)\,2}\right) = \frac{1}{b^4 T^4} & \sum_{\ell_1, k_1, \ell_2, k_2 = 1}^{[b/2]} g^\star(2\pi\ell_1/b)\, g^\star(2\pi k_1/b)\, g^\star(2\pi\ell_2/b)\, g^\star(2\pi k_2/b) \\
& \cdot \mathrm{cum}\left\{ d_T(2\pi\ell_1/b + 2\pi\tilde{i}/T) d_T(-2\pi\ell_1/b - 2\pi\tilde{i}/T) \right. \\
& \qquad \cdot d_T(2\pi k_1/b + 2\pi\tilde{i}/T) d_T(-2\pi k_1/b - 2\pi\tilde{i}/T), \\
& \qquad d_T(2\pi\ell_2/b + 2\pi\tilde{j}/T) d_T(-2\pi\ell_2/b - 2\pi\tilde{j}/T) \\
& \qquad \left. \cdot\, d_T(2\pi k_2/b + 2\pi\tilde{j}/T) d_T(-2\pi k_2/b - 2\pi\tilde{j}/T) \right\}.
\end{aligned}
$$

Correspondingly, we consider the following table:

$$
\mathcal{T}_{4,4} = \left[\begin{array}{cccc} \ell_1/b + \tilde{i}/T & -\ell_1/b - \tilde{i}/T & k_1/b + \tilde{i}/T & -k_1/b - \tilde{i}/T \\ \ell_2/b + \tilde{j}/T & -\ell_2/b - \tilde{j}/T & k_2/b + \tilde{j}/T & -k_2/b - \tilde{j}/T \end{array} \right].
$$

The types of partitions that yield the highest orders are the following: (i) four 2-sets (for a total of order T^4), (ii) two 3-sets and one 2-set (for a total of order T^3), and (iii) two 2-sets and one 4-set (for a total of order T^3). All other partitions would either involve 1-sets or involve only two sets, thereby only contributing $O(T^2)$ to the order; hence such cases are ignored.

For case (i), there are 96 indecomposable partitions yielding terms of order b^{-1}, b^{-2}, and b^{-3}, but all of which require $i = j$. For case (ii) there are even more partitions than in case (i), but the non-trivial cases contribute either $O(b^{-1}T^{-1})$ or $O(b^{-2}T^{-1})$, and these can occur when $i \neq j$. (Some of these cases occur for particular choices of $\tilde{i}$ and $\tilde{j}$, e.g., $\tilde{j} - 2\tilde{i} \in q\mathbb{Z}$ or $\tilde{i} - 2\tilde{j} \in q\mathbb{Z}$.) We do not need to be too specific about this case, since case (iii) has partitions that contribute $O(T^{-1})$, $O(b^{-1}T^{-1})$, and $O(b^{-2}T^{-1})$. The highest order T^{-1} occurs for the partitions

$$
\left[\begin{array}{cccc} \sharp & \sharp & \flat & \flat \\ \sharp & \sharp & \natural & \natural \end{array} \right] \qquad \left[\begin{array}{cccc} \sharp & \sharp & \flat & \flat \\ \natural & \natural & \sharp & \sharp \end{array} \right]
$$

$$
\left[\begin{array}{cccc} \flat & \flat & \sharp & \sharp \\ \natural & \natural & \sharp & \sharp \end{array} \right] \qquad \left[\begin{array}{cccc} \flat & \flat & \sharp & \sharp \\ \sharp & \sharp & \natural & \natural \end{array} \right]
$$

where $\sharp$ denotes membership in the 4-set, while $\flat$ and $\natural$ are for the two 2-sets. These 4 partitions have the frequencies sum to zero in each set without placing a constraint on the summation indices, so that none of the four sums collapses, thereby yielding the highest possible order of contribution asymptotically. Moreover, this analysis holds regardless of whether $i = j$. Summing over all three cases, the covariance is $O(b^{-1})$ if $i = j$ and $O(T^{-1})$ if $i \neq j$. The proof is complete. $\square$

Proof of Corollary 4.1. We proceed to verify the conditions of Corollary 3.1. Assumption B ensures that (16) holds, and thus Assumption A holds with $a_T = T^{1/2}$. The fourth moment condition on the estimator likewise follows from Assumption B; with $v = \langle g g^\star f^2 \rangle$, the convergence of the estimator's variance follows in the same way (as shown in McElroy and Roy (2022)). The stated choice of b satisfies (7), and (8) follows from the mean, variance, and covariance results of Theorem 4.1. To verify (9), write

$$\mathrm{Cov}\left\{b(\widehat{\theta}_b^{(i)}-\theta)^2, b(\widehat{\theta}_b^{(j)}-\theta)^2\right\} = b^2\mathrm{Cov}\left(\widehat{\theta}_b^{(i)\,2}, \widehat{\theta}_b^{(j)\,2}\right)$$
$$-\,2\theta b^2\mathrm{Cov}\left(\widehat{\theta}_b^{(i)\,2}, \widehat{\theta}_b^{(j)}\right) - 2\theta b^2\mathrm{Cov}\left(\widehat{\theta}_b^{(i)}, \widehat{\theta}_b^{(j)\,2}\right) + 4\theta^2 b^2\mathrm{Cov}\left(\widehat{\theta}_b^{(i)}, \widehat{\theta}_b^{(j)}\right).$$

Letting $i \neq j$, we can apply the latter two covariance results of Theorem 4.1, obtaining

$$\mathrm{Cov}\left\{b(\widehat{\theta}_b^{(i)}-\theta)^2, b(\widehat{\theta}_b^{(j)}-\theta)^2\right\} = O(b^2/T).$$

This estimate is crude, because we do not allow for the possibility of terms of the same order being cancelled out, but the resulting bound is sufficient for our purposes since $b^2/T \to 0$, thereby verifying (9). The stated result now follows from Corollary 3.1. $\square$

Proof of Theorem 4.2. Because the expectation of the skip-sample statistics tends asymptotically to θ, the mean of the skip-sample roots tends to zero. Since cumulants of all orders exist and the Gaussian distribution is characterized by its cumulants, it follows from Lemma P4.5 of Brillinger (1981) that we only need to show that the cumulants of the bivariate random variable $[S_T^{(j)}(\theta), S_T^{(k)}(\theta)]$ tend to those of a bivariate independent Gaussian. First consider the ℓth cumulant for $\ell \geq 1$:

$$\mathrm{cum}\{S_T^{(j)}(\theta), \ldots, S_T^{(j)}(\theta)\} = b^{\ell/2}\mathrm{cum}\{\widehat{\theta}_b^{(j)}, \ldots, \widehat{\theta}_b^{(j)}\}.$$

The equality follows from elementary properties of cumulants. For the $\ell = 1, 2$ cases, we know the cumulant of skip-sample statistics is asymptotic to (respectively) 0 and $b\langle gg^\star f^2\rangle$, and hence the first and second cumulants of the root tend to 0 and $\langle gg^\star f^2\rangle$ (respectively). For $\ell \geq 3$, calculations similar to those used in the proof of Theorem 4.1 show that the ℓth cumulant of the skip-sample statistic is $b^{-\ell}T^{-\ell}$ times ℓ summations of the weighting function and cumulants of various DFTs. These DFT cumulants are evaluated using a table $\mathcal{T}_{2,2,\ldots,2}$ that has 2 columns and ℓ rows. The indecomposable partitions that yield the highest order contribution are those with ℓ 2-sets, since each 2-set contributes $O(T)$. Since each 2-set must hook (i.e., it cannot be a single row of $\mathcal{T}_{2,2,\ldots,2}$, but must have each element from a different row), there are ℓ constraints on the summation indices.

Suppose that a 2-set has its elements in the same column: then it is not possible that the corresponding frequencies will sum to zero (see, e.g., the analysis of case (i) for skip-sample covariances in the proof of Theorem 4.1), and such a partition would contribute nothing to the overall cumulant. Hence, the only partitions we need consider will have the 2-sets with a member in each column (and in different rows). Consider the first 2-set, and examine its second member. A second distinct 2-set shares this row with the first 2-set. Now, the second element of the second 2-set cannot be an element of the first 2-set, for then we would have two rows consisting of just those two 2-sets, thereby violating the indecomposable property. Hence, the second element of the second 2-set shares a row with a third 2-set, and so forth (if $\ell > 3$). This analysis shows that the summation constraints induced by the indecomposable partition collapse the ℓ summations to a single summation, thereby yielding an overall order of $b^{1-\ell}$ for the skip-sample cumulant. Now multiplying by $b^{\ell/2}$, we obtain $O(b^{1-\ell/2})$ for root cumulant, which tends to zero for $\ell \geq 3$. Hence we can conclude a Gaussian limit for each root (considered univariately).

Second, to show the bivariate convergence we must consider cumulants involving ℓ occurences of $S_T^{(j)}(\theta)$ and m occurences of $S_T^{(k)}(\theta)$, where $j \neq k$. When $\ell + m \geq 3$, we can use the same arguments as in the $j = k$ case to conclude that the cumulant tends to zero. When $\ell + m = 2$, or $\ell = m = 1$, the cumulant reduces to the covariance, which is already covered in Theorem 4.1. Hence the roots are asymptotically independent and Gaussian. $\square$

Sketch of proof of Remark 4.2. The proof proceeds on analogous lines to that of Theorem 3.2. For the jth total-draw skip-sample statistic, the element of index $k_\ell^{(j)}$ is drawn from $\Lambda_{T,\ell}$, and hence

$$E\widehat{\theta}_b^{(\sharp j)} = \frac{1}{bT} \sum_{\ell=1}^{[b/2]} g^\star(2\pi\ell/b)\,(Tf(2\pi\ell/b + 2\pi\tilde{k}_\ell^{(j)}/T) + O(1)),$$

which by Taylor series expansion yields the same asymptotic result as for regular-draw skip-sampling. We remark that it is essential that in the construction of $\widehat{\theta}_b^{(\sharp j)}$, one element is drawn from each row $\Lambda_{T,\ell}$ (however, the column index $k_\ell^{(j)}$ is irrelevant); if the DFTs were drawn in such a way that they clustered, then the Taylor approximation could fail, resulting in bias.

For the covariance of total-draw skip-sample statistics, we find that

$$\begin{aligned}
\mathrm{Cov}\left(\widehat{\theta}_b^{(\sharp i)}, \widehat{\theta}_b^{(\sharp j)}\right) &= \frac{1}{b^2} \sum_{\ell_1=1}^{[b/2]} g^\star(2\pi\ell_1/b)^2\, f(2\pi\ell_1/b + 2\pi\tilde{k}_{\ell_1}^{(i)}/T)^2\, \mathbf{1}\{k_{\ell_1}^{(i)} = k_{\ell_1}^{(j)}\} \\
&+ O(b^{-1}T^{-1}) + \frac{1}{b^2 T} \sum_{\ell_1,\ell_2=1}^{[b/2]} g^\star(2\pi\ell_1/b)\, g^\star(2\pi\ell_2/b) \\
&\quad \cdot F(2\pi\ell_1/b + 2\pi\tilde{k}_{\ell_1}^{(i)}/T, -2\pi\ell_1/b - 2\pi\tilde{k}_{\ell_1}^{(i)}/T, 2\pi\ell_2/b + 2\pi\tilde{k}_{\ell_2}^{(j)}/T) + O(T^{-2}).
\end{aligned}$$

The second term involving the tri-spectral density has no constraint on the summation indices. Examination of the proof of the covariance calculations in Theorem 3.2 shows that for the two total-draw statistics the term of highest order in the covariance is $O(r/b^2)$ when $r > 0$, and otherwise is $O(1/T)$ for $r = 0$. Extending this argument to the cases of squared skip-sample statistics, we obtain either $O(r/b^2)$ or $O(1/T)$ as well.

The consistency of the total-draw skip-sampling estimator of $\widehat{v}_b$ follows in two steps: first, following the calculations of the proof of Corollary 4.1, $\mathrm{Cov}\left\{b(\widehat{\theta}_b^{(\sharp i)} - \theta)^2, b(\widehat{\theta}_b^{(\sharp j)} - \theta)^2\right\}$ is bounded by a constant times r if $r > 0$, and is $O(b^2/T)$ if $r = 0$. Second, following the calculations used in the proof of

Theorem 3.2, we find that the mean squared error of $\widetilde{v}_b$ (as defined in the proof of Corollary 3.1) is $O(1/q)+O(q/ST)+O(b/q)$. The highest order term is $O(b/q)$, but $b/q = b^2/T$, which tends to zero by the assumption that $b = o(T^{1/2})$. $\square$

Proof of Theorem 4.3. Because $g^\star = p^\star - \theta m^\star$, it follows that

$$\begin{aligned}\widehat{\theta}_b^{(j)} - \theta &= \frac{b^{-1}\sum_{\ell=1}^{[b/2]} p^\star(2\pi\ell/b)\, I_T(2\pi\ell/b + 2\pi\tilde{j}/T)}{b^{-1}\sum_{\ell=1}^{[b/2]} m^\star(2\pi\ell/b)\, I_T(2\pi\ell/b + 2\pi\tilde{j}/T)} - \theta \\ &= \frac{b^{-1}\sum_{\ell=1}^{[b/2]} g^\star(2\pi\ell/b)\, I_T(2\pi\ell/b + 2\pi\tilde{j}/T)}{b^{-1}\sum_{\ell=1}^{[b/2]} m^\star(2\pi\ell/b)\, I_T(2\pi\ell/b + 2\pi\tilde{j}/T)}.\end{aligned}$$

The denominator of the final expression, denoted by $\widehat{\langle m\, f\rangle}$ for short, converges in probability to $\langle m\, f\rangle$ by Theorem 4.1. Because θ is finite, this limit must be non-zero. Letting the numerator be denoted $\widehat{\langle g\, f\rangle}$, we obtain

$$\widehat{\theta}_b^{(j)} - \theta = \frac{\widehat{\langle g\, f\rangle}}{\langle m\, f\rangle} + \frac{\widehat{\langle g\, f\rangle}\left(\langle m\, f\rangle - \widehat{\langle m\, f\rangle}\right)}{\langle m\, f\rangle\, \widehat{\langle m\, f\rangle}}.$$

The second term has mean that is $O(b^{-1}) + O(T^{-1})$, using the Cauchy-Schwarz inequality, the delta method, and the variance results of Theorem 4.1; this is because the mean of $\widehat{\langle g\, f\rangle}$ is zero plus lower order terms. Applying Theorem 4.1 to $\widehat{\langle g\, f\rangle}/\langle m\, f\rangle$, we find that $\widehat{\theta}_b^{(j)} - E\widehat{\theta}_b^{(j)} = \widehat{\langle g\, f\rangle}/\langle m\, f\rangle + O(b^{-1}) + O(T^{-1})$. Similar calculations show that

$$\widehat{\theta}_b^{(j)\,2} - E\,\widehat{\theta}_b^{(j)\,2} = \widehat{\langle g\, f\rangle}^{\,2}/\langle m\, f\rangle^2 + O(b^{-1}) + O(T^{-1}).$$

Hence, the stated variance and covariance results follow from Theorem 4.1 applied to $\widehat{\langle g\, f\rangle}$, only noting that for the variance result we have $\langle\langle ggF\rangle\rangle = 0$, as the process is linear and $\langle g\, f\rangle = 0$. $\square$

Proof of Corollary 4.2. In view of the proof of Theorem 4.3, the proof of this corollary follows at once from Corollary 4.1. $\square$

Appendix B: Numerical Studies with Regular-draw Skip-Sampling

	First DGP		Second DGP		Third DGP		Fourth DGP	
$T = 240$	acvf	acf	acvf	acf	acvf	acf	acvf	acf
FDB Trapezoid ($h = .1$)	24.38	0.16	1.01	0.14	0.85	0.22	101.56	0.88
FDB Trapezoid ($h = .3$)	678.78	1.02	1.80	1.88	144.46	1.09	124.21	1.44
FDB Parzen ($h = .1$)	1.67	0.10	1.00	0.14	0.85	0.21	2.48	0.10
FDB Parzen ($h = .3$)	1.82	0.13	1.13	0.17	0.91	0.26	2.77	0.12
Skip ($b = 6$)	2.62	0.10	1.29	0.32	1.05	0.43	5.01	0.25
Skip ($b = 8$)	2.34	0.16	1.28	0.26	1.10	0.35	4.70	0.21
Skip ($b = 10$)	2.30	0.20	1.23	0.25	1.09	0.32	4.46	0.19
Skip ($b = 12$)	2.27	0.22	1.26	0.24	1.20	0.31	4.45	0.18
Skip ($b = 15$)	2.32	0.20	1.22	0.23	1.05	0.32	4.25	0.20
Skip ($b = 20$)	2.61	0.26	1.32	0.28	1.25	0.36	4.55	0.20
Skip ($b = 24$)	2.62	0.29	1.44	0.30	1.49	0.42	4.73	0.22
Skip ($b = 30$)	2.75	0.29	1.54	0.35	1.46	0.48	5.68	0.26
Skip ($b = 40$)	3.30	0.34	1.55	0.43	1.77	0.56	5.57	0.30
Skip ($b = 48$)	3.59	0.39	1.87	0.47	2.00	0.65	6.15	0.35
Skip ($b = 60$)	4.09	0.40	2.02	0.55	1.94	0.76	7.13	0.41
Skip ($b = 80$)	5.11	0.53	2.09	0.75	2.62	0.88	8.14	0.47

Table B.1: RMSE for Frequency Domain Bootstrap (FDB) and Regular-draw skip-sampling, for estimating the variance v of both the lag one autocovariance (acvf) and autocorrelation (acf), for four DGPs (Gaussian MA(1), Student t MA(1), Gaussian ARCH(1), and non-invertible MA(1)) of sample size $T = 240$. Bandwidth fraction for FDB is h and the size of skip-samples is b.

$T = 1200$	First DGP acvf	First DGP acf	Second DGP acvf	Second DGP acf	Third DGP acvf	Third DGP acf	Fourth DGP acvf	Fourth DGP acf
FDB Trapezoid ($h = .05$)	0.81	0.06	0.70	0.08	0.41	0.12	21.22	0.91
FDB Trapezoid ($h = .1$)	0.93	0.07	0.79	0.09	0.48	0.14	3.93	0.22
FDB Parzen ($h = .1$)	0.79	0.06	0.68	0.07	0.41	0.11	1.29	0.05
FDB Parzen ($h = .3$)	0.88	0.06	0.76	0.08	0.45	0.12	1.39	0.06
Skip ($b = 6$)	2.48	0.05	0.58	0.30	0.46	0.42	3.44	0.25
Skip ($b = 8$)	1.93	0.11	0.53	0.24	0.47	0.32	2.95	0.20
Skip ($b = 10$)	1.64	0.12	0.53	0.20	0.47	0.25	2.67	0.17
Skip ($b = 12$)	1.46	0.13	0.53	0.18	0.51	0.22	2.50	0.15
Skip ($b = 15$)	1.62	0.10	0.68	0.11	0.55	0.18	1.67	0.08
Skip ($b = 20$)	1.25	0.14	0.60	0.15	0.54	0.19	2.25	0.12
Skip ($b = 24$)	1.23	0.14	0.63	0.15	0.60	0.20	2.26	0.11
Skip ($b = 30$)	1.35	0.14	0.66	0.15	0.63	0.21	2.37	0.12
Skip ($b = 40$)	1.36	0.16	0.73	0.18	0.69	0.25	2.42	0.13
Skip ($b = 48$)	1.45	0.17	0.79	0.20	0.76	0.26	2.62	0.14
Skip ($b = 60$)	1.70	0.19	0.81	0.22	0.81	0.30	2.74	0.16
Skip ($b = 80$)	1.85	0.22	0.91	0.26	0.90	0.36	3.12	0.18

Table B.2: RMSE for Frequency Domain Bootstrap (FDB) and Regular-draw skip-sampling, for estimating the variance v of both the lag one autocovariance (acvf) and autocorrelation (acf), for four DGPs (Gaussian MA(1), Student t MA(1), Gaussian ARCH(1), and non-invertible MA(1)) of sample size $T = 1200$. Bandwidth fraction for FDB is h and the size of skip-samples is b.

Appendix C: DFT Symmetries

The DFT vector has certain symmetries; in order to describe these symmetries, we define a transposition matrix P (or P_T, when we need to annotate its dimension), which is a $T \times T$ matrix with ones on the trans-diagonal (and zeros elsewhere); its action on a vector is to reverse the order of its components. Let 1_T denote the identity matrix of dimension T, and let Π be the permutation matrix that when applied to a column vector shifts all the components upwards one position, and sends the first component to the last (bottom) position. The complex conjugate of z is denoted $\mathcal{C}z$, and the real and imaginary parts are $\Re z$ and $\Im z$ respectively.

Proposition C.1 *If T is odd, the DFT vector $\widetilde{\mathbf{X}}$ satisfies*

$$P\,\widetilde{\mathbf{X}} = \mathcal{C}\widetilde{\mathbf{X}}. \tag{C.1}$$

If T is even, the DFT vector satisfies

$$\Pi\,P\,\widetilde{\mathbf{X}} = \mathcal{C}\widetilde{\mathbf{X}}. \tag{C.2}$$

Proof of Proposition C.1. First consider the case that T is odd, so $T = 2m+1$ for some integer m. Then $P\ \widetilde{\mathbf{X}}$ has entries in reverse order, so that the middle component is unchanged but all others are flipped. Using (1), $\widetilde{X}_j = T^{-1/2}\sum_{k=1}^{T} e^{-ik\lambda_{-m-1+j}} X_k$. On the other hand, the jth component of $P\ \widetilde{\mathbf{X}}$ is $\widetilde{X}_{T+1-j} = T^{-1/2}\sum_{k=1}^{T} e^{-ik\lambda_{m+1-j}} X_k$, which is the conjugate of $\widetilde{X}_j$. This proves (C.1). Next, suppose that T is even, so $T = 2m$ for some integer m. For $j = 1, \ldots, T-1$, the jth component of $\Pi\,P\,\widetilde{\mathbf{X}}$ is the $j+1$th component of $P\,\widetilde{\mathbf{X}}$, which is

$$\widetilde{X}_{T-j} = T^{-1/2}\sum_{k=1}^{T} e^{-ik\lambda_{m-j}} X_k = T^{-1/2}\sum_{k=1}^{T} e^{ik\lambda_{-m+j}} X_k = \mathcal{C}\widetilde{X}_j.$$

Moreover, the Tth component of $\Pi\,P\,\widetilde{\mathbf{X}}$ is the first component of $P\,\widetilde{\mathbf{X}}$, i.e., $\widetilde{X}_T$. Because $\lambda_m = 2\pi m/T = \pi$, this number is real, and hence $\widetilde{X}_T = \mathcal{C}\widetilde{X}_T$. This proves (C.2). $\square$

In view of Proposition C.1, we say that the DFT vector satisfies a Symmetry Property, defined as follows.

Definition C.1 *A length T complex vector $\widetilde{\mathbf{X}}$ satisfies the Symmetry Property if and only if (C.1) holds when T is odd and (C.2) holds when T is even.*

A general length T complex vector $\mathbf{Z}$ may satisfy the Symmetry Property, in which case necessarily the vector has a particular structure. If $\mathbf{Z}$ satisfies (C.1), it must be the case that

$$\Re[P\mathbf{Z}] = \Re[\mathbf{Z}], \qquad \Im[P\mathbf{Z}] = -\Im[\mathbf{Z}].$$

Because the middle entry of an odd-length $\mathbf{Z}$ has its value unchanged after application of P, its value must be real. Similarly, a vector $\mathbf{Z}$ satisfying (C.2) has real entries for components $T/2$ and T, whereas the subvector of components 1 through $T/2-1$ is the complex conjugate of the transposition of the subvector for components $T/2+1$ through $T-1$. In this paper we will be constructing DFT vectors in various ways, but we need to ensure that these constructions have the correct properties. In particular, given a complex vector $\mathbf{Z}$ it behooves us to know how it can be modified so as to have the Symmetry Property. The following proposition justifies this motivation.

Proposition C.2 *If $\mathbf{Z}$ has the Symmetry Property, then $Q\mathbf{Z}$ has real-valued entries.*

Proof of Proposition C.2. First, noting that $P_{jk} = \mathbf{1}\{j+k = T+1\}$, we obtain

$$\begin{aligned}
\{\mathcal{C}QP\}_{jk} &= \sum_{\ell=1}^{T} \mathcal{C}Q_{j\ell}\, P_{\ell k} = \overline{Q}_{j,T+1-k} \\
&= T^{-1/2}\, \exp\{-2\pi i j([T/2] - T + T + 1 - k)/T\} \\
&= T^{-1/2}\, \exp\{2\pi i j(k - 1 - [T/2])/T\}.
\end{aligned}$$

If T is odd, then $-1-[T/2] = [T/2]-T$, and $\{\mathcal{C}[QP]\}_{jk} = Q_{jk}$. But if T is even, then $-[T/2] = [T/2]-T$ and $\{\mathcal{C}[QP]\}_{jk} = Q_{j,k-1}$ for $k = 2, \ldots, T$; also $\{\mathcal{C}[QP]\}_{j1} = Q_{j,T}$ because $\exp\{2\pi i j([T/2]-T)/T\} = \exp\{2\pi i j([T/2])/T\}$. Therefore, when T is odd $\mathcal{C}[QP] = Q$, but when T is even $\mathcal{C}[QP] = Q\Pi$. Next, because P is idempotent

$$\mathcal{C}[Q\,\mathbf{Z}] = \mathcal{C}[Q]P\,P\mathcal{C}[\mathbf{Z}] = \begin{cases} Q\,P\mathcal{C}[\mathbf{Z}] & \text{if } T \text{ is odd} \\ Q\,\Pi P\mathcal{C}[\mathbf{Z}] & \text{if } T \text{ is even} \end{cases} = Q\mathbf{Z},$$

using (C.1) and (C.2). Hence $Q\mathbf{Z}$ equals its own conjugate, and therefore must be real. $\square$

Remark C.1 As an application, we can alter a given complex vector $\mathbf{Z}$ to have the Symmetry Property as follows. If the length T is odd, replace the first $[T/2]$ entries with the conjugate of the first $[T/2]$ entries of $P\mathbf{Z}$, and discard the imaginary part of the middle entry in position $[T/2]+1$. If the length of T is even, we replace components 1 through $T/2-1$ with the conjugate of components $T-1$ through $T/2+1$ (so their order is flipped); also, the imaginary portions of components $T/2$ and T are discarded. These operations ensure that the modified $\mathbf{Z}$ has the Symmetry Property.