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Locally Stationary Processes and the Local Block Bootstrap

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For time series that are not stationary, the block bootstrap of Künsch (1989) is not directly applicable. However, if the underlying stochastic structure is slowly changing with time, one may employ the local block bootstrap of Paparoditis and Politis (2002). We describe the local block bootstrap procedure, and show its consistency in an interesting example of a locally stationary process.

1. INTRODUCTION

Let $Y_1, ..., Y_n$ be an observed stretch of a real-valued time series $\{Y_t, t \in \mathbf{Z}\}$. In many applications, for instance in connection with financial or meteorological time series, the sample size n may be quite large. Consequently, it may be unrealistic to assume that such a long time series is stationary; a more realistic assumption is that of a slowly-changing stochastic structure in the sense that the joint probability law of $(Y_t, Y_{t+1}, \ldots, Y_{t+k})$ changes smoothly (and slowly) with t for any k. Such nonstationary models have been provided by the evolutionary spectra models of Priestley (1988) and the locally stationary models of Dahlhaus (1996, 1997).

Now because of the nonstationarity of $\{Y_t\}$, the block bootstrap method of Künsch (1989) is not directly applicable, and neither are its different variations such as the stationary bootstrap; see e.g. Politis (2003) for a review. Rather, a modification that takes into account the changing stochastic structure should be constructed. Such a modification has been proposed in the Local Block Bootstrap (LBB) procedure of Paparoditis and Politis (2002). The basic premise of the LBB is to only resample blocks that are close to each other, i.e., a block that starts at time t, can only be replaced with blocks whose starting point is close to t. The LBB idea is to construct a bootstrap pseudo-series by a concatenation of q blocks of size b (such that $qb \simeq n$), where the jth block of the resampled series is chosen randomly from a distribution (say, uniform) on all the size-b blocks of consecutive data whose time indices are 'close' to those in the original block.

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A more detailed description of the LBB algorithm is now presented. Given the data $Y_1, \dots, Y_n$, the LBB algorithm creates a bootstrap pseudo-series $Y_1^*, \dots, Y_n^*$ as follows:

- Select an integer block size b, and a real number $B \in (0,1]$ such that nB is an integer; both b and B are thought of as functions of n.
- For $h = 0, 1, \dots, (\lceil n/b \rceil - 1)$, let $Y^*_{hb+j} := Y_{I_h+j-1}$ for $j = 1, \dots, b$, where $I_1, I_2, \dots$ are independent, integer-valued random variables satisfying $P(I_h = k) = W_{n,h}(k)$.

Here $\lceil k \rceil$ denotes the smallest integer that is greater or equal to k. Thus, in the case where n is not an integer multiple of b, the LBB algorithm generates a bootstrap series whose length is slightly bigger than n, although only $Y_1^*, \dots, Y_n^*$ are kept in the end.

The probability distribution $W_{n,h}(k)$ is the practitioner's choice. The easiest choice is the uniform probability over the integers in the interval $[J_{1,h}, J_{2,h}]$, where $J_{1,h} = \max\{1, hb - nB\}$ and $J_{2,h} = \min\{n - b + 1, hb + nB\}$. We will adopt this simple choice in the sequel although many other choices are possible; see Paparoditis and Politis (2002).

For an arbitrary point k, the range $[k - nB, k + nB]$ indicates its 'local stationarity' neighborhood; in other words, although the time series $\{X_t\}$ is not (globally) stationary, the stretch $X_{k-nB}, \dots, X_{k+nB}$ can be thought to have been generated by an approximate stationary mechanism provided that the local window size $2nB$ is small with respect to the sample size n. Nevertheless, we would also need $nB \to \infty$ so that enough data accumulate in the local window. As a matter of fact, doing a block bootstrap within the local window would require a block size b that is big but small with respect to the local window size $2nB$.

The required interplay between block and window size is typically given by a requirement such as eq. (2). However, the rates in eq. (2) are specific to the problem addressed in the next section, i.e., showing the consistency of the LBB in the context of data from the particular locally stationary process (1) below; all technical proofs are placed in the Appendix.

2. CONSISTENCY OF THE LOCAL BLOCK BOOTSTRAP

As is well-known, the applicability of bootstrap methods is usually checked on a case-by-case basis; the only exception so far seems to be the i.i.d. bootstrap with smaller resample size that is generally consistent; see e.g. Politis, Romano and Wolf (1999). Thus, we now focus on a particular interesting example of a locally stationary process in the sense of Dahlhaus (1996, 1997). Let the data be generated from the model[2]

$$Y_t = s_n(t) + \mu + v_n(t)e_t, \quad \text{for all } t \in \mathbf{Z} \tag{1}$$

where $\{e_t, t \in \mathbf{Z}\}$ is a mean-zero and variance-one, strong mixing and strictly stationary sequence satisfying:

$$E|e_t|^{6+\delta} < \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} k^2 \alpha^{\delta/(6+\delta)}(k) < \infty \quad \text{for some } \delta > 0.$$

[2]Note that under (1), the data $Y_1, \dots, Y_n$ constitute the nth row of a triangular array; however, since no confusion arises, we will not use the usual double-index notation.

where $\alpha(k)$ indicates the strong mixing coefficient associated with $\{e_t, t \in \mathbf{Z}\}$. We also assume that $s_n(t) = m(t/n)$, $v_n(t) = \sigma(t/n)$, for some fixed functions m, σ that are differentiable[3] with bounded derivatives on [0,1]. We will further assume that $\sigma(x) > 0$ for all x in [0,1] and that m satisfies $\int_0^1 m(x)dx = 0$; due the latter, m may be thought of as a 'seasonality' fluctuation about the 'grand mean' μ.

Our goal is interval estimation of the unknown parameter μ based on the data $Y_1, \dots, Y_n$. For this reason we require an approximation to the sampling distribution of the sample mean $\hat{\mu} = n^{-1}\sum_{t=1}^n Y_t$ that will serve as our estimator of μ. We propose the LBB as a method for this approximation.

We first establish some of the properties of $\hat{\mu}$. The following two lemmas concern the asymptotic bias and variance of the sample mean.

Lemma 1. $E\left(n^{\frac{1}{2}}(\hat{\mu} - \mu)\right) = O(n^{-\frac{1}{2}})$.

Lemma 2. Let $c(s) = Cov(e_0, e_s)$. Then, as $n \to \infty$, $Var\left(n^{\frac{1}{2}}(\hat{\mu} - \mu)\right) \to V^2$, where $V^2 = \sum_{s=-\infty}^{\infty} c(s) \int_0^1 \sigma^2(u)du$.

In the next theorem we establish asymptotic normality of the sample mean.

Theorem 1. As $n \to \infty$, $\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{\mathcal{L}} N(0, V^2)$.

We now turn to the LBB properties. Let $Y_1^*, \dots, Y_n^*$ denote an LBB pseudo-series constructed using the algorithm of the previous section, and let $\hat{\mu}^* = n^{-1}\sum_{t=1}^n Y_t^*$ be the bootstrap sample mean. As usual, let P^*, E^*, Var^* indicate probability, expectation, and variance under the LBB scheme (conditional on the data $Y_1, \dots, Y_n$).

To investigate the consistency of the LBB we will—as previously mentioned—require some conditions on the block size, as well as the window size indicating the local neighborhood. For this reason, consider the following:

$$\text{Let } n \to \infty,\ b \to \infty \ \text{ but } \ b = o\left(\min(n^{\frac{1}{2}}, nB)\right) \to 0, \ \text{ and } \ nB \to \infty \ \text{ but } \ nB^2 \to 0. \tag{2}$$

Lemma 3 below can be compared to Lemma 1 and 2.

Lemma 3. Under (2), $E^*\left(n^{\frac{1}{2}}(\hat{\mu}^* - \hat{\mu})\right) = O_p(nB^2)$, and $Var^*\left(n^{\frac{1}{2}}(\hat{\mu}^* - \hat{\mu})\right) \xrightarrow{P} V^2$.

Our main theorem below shows that the LBB is successful in giving a consistent approximation to the sampling distribution of the sample mean $\hat{\mu}$.

Theorem 2. Under (2), we have

$$\sup_x |P(\sqrt{n}(\hat{\mu} - \mu) \le x) - P^*(\sqrt{n}(\hat{\mu}^* - E^*\hat{\mu}^*) \le x)| \xrightarrow{P} 0, \tag{3}$$

as well as

$$\sup_x |P(\sqrt{n}(\hat{\mu} - \mu) \le x) - P^*(\sqrt{n}(\hat{\mu}^* - \hat{\mu}) \le x)| \xrightarrow{P} 0. \tag{4}$$

[3]Under slightly more careful arguments, the conditions on m and σ may be relaxed to just piecewise Lipschitz continuity as suggested in Paparoditis and Politis (2002).

From Theorem 2 it follows that asymptotically valid confidence intervals for μ can be based on the quantiles of either the bootstrap distribution $P^*(\sqrt{n}(\hat{\mu}^* - E^*\hat{\mu}^*) \le x)$ or the bootstrap distribution $P^*(\sqrt{n}(\hat{\mu}^* - \hat{\mu}) \le x)$, both of which are computable—as opposed to the quantiles of the unknown true distribution $P(\sqrt{n}(\hat{\mu} - \mu) \le x)$.

To help delineate which of these two approximations may be preferrable, recall that the distribution $P^*(\sqrt{n}(\hat{\mu}^* - \hat{\mu}) \le x)$ has center of location $O_p(nB^2)$ by Lemma 3, while $P^*(\sqrt{n}(\hat{\mu}^* - E^*\hat{\mu}^*) \le x)$ has center exactly zero, and $P(\sqrt{n}(\hat{\mu} - \mu) \le x)$ has center $O(1/\sqrt{n})$ by Lemma 1. In order to satisfy (2) we may let $B = 1/n^{\epsilon+1/2}$ for some ϵ in $(0, 1/2)$. Typically, we may even have $\epsilon < 1/4$, in which case approximation (3) is preferrable as its (zero) center of location is closest to that of the target distribution $P(\sqrt{n}(\hat{\mu} - \mu) \le x)$. In addition, the validity of (3) may be proved under slightly weaker conditions, namely replacing the condition $nB^2 \to 0$ in (2) by the weaker $B \to 0$.

3. APPENDIX: TECHNICAL PROOFS

Proof of Lemma 1.

$$E\left(n^{\frac{1}{2}}(\hat{\mu} - \mu)\right) = E\left(n^{\frac{1}{2}}\left(\left(\frac{1}{n}\sum_{t=1}^{n}(s_n(t) + \mu + v_n(t)e_t)\right) - \mu\right)\right)$$

$$= E\left(n^{\frac{1}{2}}\left(\frac{1}{n}\sum_{t=1}^{n} s_n(t) + \frac{1}{n}\sum_{t=1}^{n} v_n(t)e_t\right)\right)$$

$$= n^{\frac{1}{2}}\left(\frac{1}{n}\sum_{t=1}^{n} s_n(t) + \frac{1}{n}\sum_{t=1}^{n} v_n(t)E(e_t)\right)$$

$$= n^{\frac{1}{2}}\left(\frac{1}{n}\sum_{t=1}^{n} m(t/n)\right) = n^{\frac{1}{2}}O\left(\frac{1}{n}\right) = O(n^{-\frac{1}{2}})$$

since $\frac{1}{n}\sum_{t=1}^{n} m(t/n)$ is the Riemann sum approximation of the integral $\int_0^1 m(u)du = 0$.

Proof of Lemma 2.

$$Var\left(n^{\frac{1}{2}}(\hat{\mu} - \mu)\right) = nVar\left(\frac{1}{n}\sum_{t=1}^{n} v_n(t)e_t\right)$$

$$= \frac{1}{n}Var\left(\sum_{t=1}^{n}\sigma(\frac{t}{n})e_t\right) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma(\frac{i}{n})\sigma(\frac{j}{n})c(i-j)$$

$$= \frac{1}{n}\sum_{s=-n+1}^{n-1}\sum_{r=1}^{n-|s|}\sigma(\frac{r}{n})\sigma(\frac{r+|s|}{n})c(s) \tag{5}$$

Using the Mean Value Theorem and letting $\xi_n^{r,s} \in [\frac{r}{n}, \frac{r+|s|}{n}]$ we can re-write (5) as

$$\frac{1}{n}\sum_{s=-n+1}^{n-1}\sum_{r=1}^{n-|s|}\sigma(\frac{r}{n})\left(\sigma(\frac{r}{n}) + \frac{|s|}{n}\sigma'(\xi_n^{r,s})\right)c(s)$$

$$= \frac{1}{n}\sum_{s=-n+1}^{n-1}\sum_{r=1}^{n-|s|}\left(\sigma(\frac{r}{n})\right)^2 c(s) + \frac{1}{n}\sum_{s=-n+1}^{n-1}\sum_{r=1}^{n-|s|}\sigma(\frac{r}{n})\frac{|s|}{n}\sigma'(\xi_n^{r,s})c(s)$$

$= V_1 + V_2$ respectively.

Let us consider $|V_2|$. We have

$$\left|\frac{1}{n}\sum_{s=-n+1}^{n-1}\sum_{r=1}^{n-|s|}\sigma(\frac{r}{n})\frac{|s|}{n}\sigma'(\xi_n^{r,s})c(s)\right|$$

$$\le \frac{C}{n}\sum_{s=-n+1}^{n-1}\sum_{r=1}^{n-|s|}\frac{|s|}{n}|c(s)| \le \frac{C}{n}\sum_{s=-n+1}^{n-1}\sum_{r=1}^{n}\frac{|s|}{n}|c(s)|$$

$$= O\left(\sum_{s=-n+1}^{n-1}\frac{|s|}{n}|c(s)|\right) = o(1)$$

where $C = \max_{x,y\in[0,1]}|\sigma(x)||\sigma'(y)|$. The limit is achieved by the mixing inequalities and Kronecker's Lemma.

We can write V_1 as

$$\frac{1}{n}\sum_{s=-n+1}^{n-1}\sum_{r=1}^{n}\left(\sigma(\frac{r}{n})\right)^2 c(s) - \frac{1}{n}\sum_{s=-n+1}^{n-1}\sum_{r=n-|s|+1}^{n}\left(\sigma(\frac{r}{n})\right)^2 c(s)$$

which can be written as $V_1' - V_1''$. We have that

$$|V_1''| = |\frac{1}{n}\sum_{s=-n+1}^{n-1}\sum_{r=n-|s|+1}^{n}\left(\sigma(\frac{r}{n})\right)^2 c(s)| \le C_1\frac{1}{n}\sum_{s=-n+1}^{n-1}|s||c(s)| \to 0$$

where $C_1 = \max_{x\in[0,1]}(\sigma(x))^2$. Finally,

$$V_1' = \sum_{s=-n+1}^{n-1} c(s)\frac{1}{n}\sum_{r=1}^{n}\left(\sigma(\frac{r}{n})\right)^2 \to \sum_{s=-\infty}^{\infty} c(s)\int_0^1 \sigma^2(u)du$$

and the lemma is proven.

Proof of Theorem 1. Observe that we have a weighted sum of strong mixing random variables where the weights obey the conditions of Theorem 3.1 of Roussas, Tran and Ioannides (1992) –RTI for short– on fixed design regression. The problem is to show that $\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sigma_t e_t$ is asymptotically normally distributed under our assumptions. To do that, we need to verify the assumptions and conditions in RTI. For example, (A1) of RTI is satisfied by the definition of our model along with our assumptions. (A2) (i) of RTI is a consequence of the fact that

$$\frac{1}{n}\sum_{i=1}^{n}\sigma(\frac{i}{n}) \to C_0 \text{ as } n \to \infty.$$

for some constant $C_0 > 0$ since σ is a positive integrable function. (A2) (ii) of RTI is a result of the fact that there exists a constant $C > 0$ such that $\sigma(u) < C$ uniformly in u. Furthermore,

$$\sum_{i=1}^{n}\left(\frac{\sigma(i/n)}{n}\right)^2 = \frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^{n}\sigma(\tfrac{i}{n})^2\right) = O\left(\frac{1}{n}\right)$$

because

$$\frac{1}{n}\sum_{i=1}^{n}\sigma(\tfrac{i}{n})^2 \to C_1 \text{ as } n \to \infty$$

for some $C_1 > 0$. Therefore we have

$$\max_i\left|\frac{\sigma(\frac{i}{n})}{n}\right| = O\left(\sum_{i=1}^{n}\left(\frac{\sigma(\frac{i}{n})}{n}\right)^2\right).$$

Assumption (A3) of RTI is satisfied by the fact that the variance of $\hat{\mu}$ is $O(\frac{1}{n})$ by Lemma 2 which implies that

$$\sum_{i=1}^{n}\left(\frac{\sigma(\frac{i}{n})}{n}\right)^2 = O(Var(\hat{\mu})).$$

Assumption (A4) of RTI is an immediate consequence of our assumptions on the sequence e_t. Condition (2.21) of RTI has three parts. We first observe that the effective number of terms in our sum is n. We have to check that there exist p and q as defined in RTI where $qp^{-1} \to 0$ and where

$$nqp^{-1}\sum_{i=1}^{n}\left(\frac{\sigma(\frac{i}{n})}{n}\right)^2 \to 0.$$

$$p^2\sum_{i=1}^{n}\left(\frac{\sigma(\frac{i}{n})}{n}\right)^2 \to 0 \quad \text{and} \quad nqp^{-1}\alpha(q) \to 0 \text{ as } n \to \infty$$

since as shown earlier

$$n\sum_{i=1}^{n}\left(\frac{\sigma(\frac{i}{n})}{n}\right)^2 = O(1),$$

a possible choice is $p = n^{\frac{1}{2}-\varepsilon}$ and $q = n^{\frac{1}{4}+\varepsilon}$ for some $\varepsilon \in (0, 1/4)$. As a result we confirm that the first part of condition (2.21) of RTI holds. Finally, from our mixing assumption on e_t we have that $\alpha(q) = o(1/q^2)$ and therefore $nqp^{-1}\alpha(q) = o(1)$ which satisfies the last part of condition (2.21) of RTI and the theorem is proven.

Proof of Lemma 3. The proof is rather long and tedious. We give just the main idea below for lack of space and refer the reader to Dowla (2002) for details. Consider the effect of the LBB on the underlying quantities in eq. (1), namely $m(\frac{t}{n})$, $\mu, \sigma(\frac{t}{n})$ and e_t. The LBB actually performs a similar re-arrangement to those unobservable quantities as the one

performed on the data Y_t. In other words, for $h = 0, 1, \ldots, ([n/b] - 1)$ and $j = 1, \ldots, b$, we have $m^*(\frac{hb+j}{n}) = m(\frac{I_h+j-1}{n})$, $\sigma^*(\frac{hb+j}{n}) = \sigma(\frac{I_h+j-1}{n})$, $\mu^* = \mu$, and $e^*_{hb+j} = e_{I_h+j-1}$, where $I_1, I_2, \ldots$ are the *same* values used to generate Y_t^*.

Therefore, we have the LBB bootstrap version of eq. (1) given below.

$$Y_t^* = m^*(\frac{t}{n}) + \mu + \sigma^*(\frac{t}{n})e_t^*. \tag{6}$$

The key observation now is that $m^*(\frac{t}{n}) - m(\frac{t}{n}) = O(B)$ uniformly in t by Taylor's theorem; thus, $n^{-1}\sum_{t=1}^{N}(m^*(\frac{t}{n}) - m(\frac{t}{n})) = O(B)$ as well. Similarly, $\sigma^*(\frac{t}{n}) - \sigma(\frac{t}{n}) = O(B)$ uniformly in t; recall that $B \to 0$ by eq. (2).

Thus, an expression of the type $E^*\left(n^{\frac{1}{2}}(\hat{\mu}^* - \hat{\mu})\right)$ and/or $Var^*\left(n^{\frac{1}{2}}(\hat{\mu}^* - \hat{\mu})\right)$ can be significantly simplified since $\hat{\mu}^*$ is an average of (a linear combination of) the quantities $m^*(\frac{t}{n}), \sigma^*(\frac{t}{n})$ and e_t^* that are closely related to the quantities $m(\frac{t}{n}), \sigma(\frac{t}{n})$ and e_t found within $\hat{\mu}$. More details can be found in Dowla (2002, Ch. 4). At the final analysis, the LBB on the errors e_t is tantamount to (a version of) the regular block bootstrap for stationary sequences that is consistent for the variance of the sample mean.

Proof of Theorem 2. Consider the expression

$$n^{\frac{1}{2}}(\hat{\mu}^* - E^*\hat{\mu}^*) = n^{\frac{1}{2}}\left(\sum_{i=1}^{n}\frac{1}{n}(Y_i^* - E^*Y_i^*)\right).$$

We can rewrite the above in terms of a sum of blocks of size b, i.e.,

$$\sum_{i=0}^{\lfloor\frac{n}{b}\rfloor-1}\left(n^{-\frac{1}{2}}\sum_{j=1}^{b}(Y^*_{ib+j} - E^*Y^*_{ib+j})\right) = \sum_{i=0}^{\lfloor\frac{n}{b}\rfloor-1}\xi^*_{i,n}.$$

Note that the random variable $\xi^*_{i,n}$ is a function of the i^{th} independent block of data of size b. We need to show that the $\xi^*_{i,n}$'s satisfy the Central Limit Theorem for a triangular array sum of independent, non-identically distributed random variables. Since we have finite moments of order 6+δ we can use the Lyapunov condition. From Lemma 3, and the independence of the $\xi^*_{i,n}$'s, it follows that

$$\sum_{i=0}^{\lfloor\frac{n}{b}\rfloor-1} Var^*(\xi^*_{i,n}) \xrightarrow{P} \sum_{s=-\infty}^{\infty} c(s)\int_0^1 \sigma^2(u)du.$$

We also have

$$E^*(\xi^{*(6)}_{i,n}) = E^*\left(n^{-\frac{1}{2}}\sum_{j=1}^{b}(Y^*_{ib+j} - E^*Y^*_{ib+j})\right)^6 = \frac{1}{n^3}O_p(b^6)$$

Therefore we obtain

$$\frac{\sum_{i=0}^{\lfloor\frac{n}{b}\rfloor-1} E^*(\xi^{*6}_{i,n})}{\left(\sum_{i=0}^{\lfloor\frac{n}{b}\rfloor-1} Var^*(\xi^*_{i,n})\right)^2} = \frac{\frac{n}{b}\left(\frac{1}{n^3}O_p(b^6)\right)}{\left(\sum_{s=-\infty}^{\infty} c(s)\int_0^1\sigma^2(u)du\right)^2} = O_p\left(\frac{b^5}{n^2}\right)$$

Using (2), we can invoke Lyapunov's condition to establish that—over an event whose probability tends to one—the random variable $n^{\frac{1}{2}}(\hat{\mu}^* - E^*\hat{\mu}^*)$ is asymptotically normal with mean zero. Furthermore, from Lemma 3 we know that $\hat{\mu}^*$ has the correct asymptotic variance. Thus, the theorem is proven.

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11. Time Series and Stochastic Processes