Non-parametric Sequential Estimation of a Regression Function Based on Dependent Observations

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Abstract: This paper presents a sequential estimation procedure for an unknown regression function. Observed regressors and noises of the model are supposed to be dependent and form sequences of dependent numbers. Two types of estimators are considered. Both estimators are constructed on the basis of Nadaraya–Watson kernel estimators.

First, sequential estimators with given bias and mean square error are defined. According to the sequential approach the duration of observations is a special stopping time. Then on the basis of these estimators of a regression function, truncated sequential estimators on a time interval of a fixed length are constructed. At the same time the variance of these estimators is controlled by a (non-asymptotic) bound.

In addition to non-asymptotic properties, the limiting behavior of presented estimators is investigated. It is shown, in particular, that by the appropriate chosen bandwidths both estimators have optimal (as compared to the case of independent data) rates of convergence of Nadaraya–Watson kernel estimators.

Keywords: Non-parametric kernel regression estimation; Dependent observations; Sequential approach; Guaranteed mean square accuracy; Finite sample size.

Subject Classifications: 62G05; 62G08; 62G20; 62J02; 62L12.

1. INTRODUCTION

This paper considers the problem of non-parametric kernel estimation of a regression function $f(x)$, $x \in \mathbb{R}^1$ at a point $x = x^0$ satisfying the equation

$$Y_i = f(x_i) + \delta_i, \quad i \geq 1, \quad (1.1)$$

in the case of dependent regressors $X = (x_i)_{i \geq 1}$ and noises $\Delta = (\delta_i)_{i \geq 1}$. It is supposed, that pairs (Y_i, x_i) , $i = 1, 2, \dots$ are under observations but the errors δ_i are unobservable. We allow in the paper for general dependence conditions on random variables (x_i) and (δ_i) supposing mutual dependence of the processes X and Δ . In particular, model inputs can be unbounded with a positive probability and estimation problem of non-linear autoregressive function can be solved (see Section 5.2).

There are a lot of results and publications dedicate this problem for independent and dependent observations, in asymptotic and non-asymptotic problem statements (see, e.g. Bai and Liu (2007), Dharmasena at al. (2008)-Honda (1998), Leonov (1999), Nazin and Ljung (2002), Politis at al. (1999)-Scott (1992) etc.) It should be noted, that often regressors X a supposed to be non-random or bounded and noises Δ of the model are independent, see Bai and Liu (2007); Dharmasena at al. (2008); Györfi at al. (2002); Nazin and Ljung (2002); Politis at al. (1999), Roll at al. (2002)-Rosenblatt (1991) etc.

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The main goal of the paper is twofold. We will construct sequential and so-called truncated sequential estimators of a regression function. Estimators of both types have known mean square errors (MSE's). To obtain sequential estimators with an arbitrary accuracy one needs to have a sample of unbounded size. This approach has been primarily proposed by A. Wald for various statistical problems for a scheme of independent observations, see Wald (1947, 1950). It proposes a construction of a special stopping time as a duration of observations to obtain statistics with desired properties. Then this idea has been applied to parameter estimation problem of one-parameter stochastic differential equations in Liptser and Shiryaev (1978); Novikov (1971) and for multiparameter continuous and discrete-time dynamic systems in many papers and books (see Borisov and Konev (1977); Dobrovidov et al. (2012); Konev (1985); Kchler and Vasiliev (2010) among others). Sequential approach has also been applied to non-parametric regression and density function estimation problems as well (see, e.g., Carroll (1976); Davies and Wegman (1975), Dharmasena et al. (2008)-Efroimovich (2007), Hondaa (1998); Liski et al. (2002); Martinsek (1992); Mukhopadhyay and Solanky (1994); Rao (1983); Stute (1983); Vasiliev (1997); Vasiliev and Koshkin (1998); Xu and Martinsek (1995)). Peculiarities and difficulties of sequential approach to nonparametric problems are discussed in detail in Efroimovich (2007).

However in practice the observation time of a system is usually not only finite but fixed which does not allow for the use of such estimators. One of the possibilities for finding estimators with the guaranteed quality of inference using a sample of fixed size is provided by the approach of truncated sequential estimation. Truncated sequential estimation method was developed in Konev and Pergamenshchikov (1990a)-Konev and Pergamenshchikov (1992), as well as in Fourdrinier et al. (2009) for parameter estimation problems in discrete and continuous time dynamic models. Using sequential approach, they have constructed estimators of dynamic systems parameters with known variance by samples of fixed size.

Non-parametric truncated sequential estimators of a regression function presented in this paper constitute Nadaraya–Watson estimators calculated at a special stopping time. These estimators have known mean square errors as well. The duration of observations is also random but bounded from above to a non-random fixed number.

Non-asymptotic and asymptotic properties of estimators of both types are investigated. It is shown, that truncated sequential estimators coincide with the afore-mentioned sequential estimators (i.e. with the Nadaraya–Watson estimators calculated at a special stopping time) for sufficiently large sample size.

It should be noted, that usually non-parametric estimators of functionals are studied by asymptotic assumptions even in sequential problem statement (except of some special cases, see, e.g., Dobrovidov et al. (2012) and references therein). Nadaraya–Watson estimators of a regression function with given non-asymptotic properties are derived here for the first time.

The outline of the paper is as follows. In Section 2 basic assumptions on a regression function to be estimated, as well as regressors and noises of the model are given. In Section 3 we present a construction and basic properties of sequential estimation procedure of the regression function, which can be estimated with an arbitrary mean square accuracy. Section 4 presents truncated sequential estimation procedure, which give a possibility to obtain an estimator of the regression function with a known variance based on a sample of finite size. In Section 5 we consider basic examples of dependence type of regressors and noises of the model. In particular, non-linear autoregressive estimation problem is considered. Section 6 is the conclusion of the paper. In Section 7 all proves are given.

Similar results were presented in Politis and Vasiliev (2012).

2. PROBLEM SET-UP AND BASIC ASSUMPTIONS

Consider the problem of non-parametric estimation of a regression function $f(x)$ at a point $x = x^0$ from the equation (1.1) based on data (Y_i, x_i) , $i = 1, 2, \dots$.

The notation $\overline{1, m}$ means the set of integers $1, 2, \dots, m$.

We suppose that the function to be estimated satisfies the following assumption.

Assumption (f): The function $f(x)$ is p -th differentiable at a point x^0 , $p \geq 0$ and for some positive numbers $C_{f,i}$, $i = \overline{0, p}$ ($p \geq 0$), $L_f, \gamma_f \in (0, 1]$, such that

$$|f^{(i)}(x^0)| \leq C_{f,i}, \quad i = \overline{0, p}$$

and for every $x, y \in \mathbb{R}^1$,

$$|f^{(p)}(x) - f^{(p)}(y)| \leq L_f |x - y|^{\gamma_f}.$$

For estimation of the function $f(\cdot)$ at a point x^0 , $f = f(x^0)$ we use so-called sequential estimators, constructed on the basis of kernel type estimators of the form

$$\hat{f}_N = \left(\sum_{i=1}^N K \left(\frac{x^0 - x_i}{h} \right) \right)^+ \cdot \sum_{i=1}^N Y_i K \left(\frac{x^0 - x_i}{h} \right), \quad N \geq 1, \quad (2.1)$$

where $a^+ = a^{-1}$ for $a \neq 0$; and $a^+ = 0$ when $a = 0$; $K(\cdot)$ is a kernel function and h is a bandwidth, satisfying the Assumptions (K) and (h) below respectively.

Define for every $i \geq 1$ density functions $f_{i-1}(t)$ of conditional distribution functions $F_{i-1}(t) = P(x_i \leq t | \mathcal{F}_{i-1}^x)$, $\mathcal{F}_i^x = \sigma\{x_1, \dots, x_i\}$.

As regards to the regressors X and Δ we suppose

Assumption (X): Assume the conditional density functions $f_{i-1}(t)$, $i \geq 1$ are q -th differentiable at a point x^0 , $q \geq 0$ and for some positive numbers c_x^0 , $C_{x,k}$, $k = \overline{0, q}$ ($q \geq 0$), L_x and $\gamma_x \in (0, 1]$ the following relations

$$\inf_{i \geq 1} f_{i-1}(x^0) \geq c_x^0,$$

$$\sup_{i \geq 1} |f_{i-1}^{(k)}(x^0)| \leq C_{x,k}, \quad k = \overline{0, q}$$

and for every $x, y \in \mathbb{R}^1$,

$$\sup_{i \geq 1} |f_{i-1}^{(q)}(x) - f_{i-1}^{(q)}(y)| \leq L_x |x - y|^{\gamma_x}$$

are fulfilled.

Assumption (Δ): Let the noises Δ be conditionally zero mean $E(\delta_i | \mathcal{F}_i^x) = 0$, $i \geq 1$ and for some monotonically non-increasing as $k \rightarrow \infty$ non-negative function $\varphi(k)$, $k \geq 1$ the following "general mixing condition" for the sequence Δ holds true

$$|E(\delta_i \delta_j | \mathcal{F}_{i \vee j}^x)| \leq \varphi(|i - j|), \quad i, j \geq 1.$$

Define admissible sets of kernels $K(\cdot)$ and bandwidths h as follows.

Assumption (K): Let the function $K(\cdot)$ is uniformly bounded from above:

$$\sup_z |K(z)| \leq \overline{K} < \infty$$

and such that

$$\int K(z)dz = 1, \quad \int |z^s K(z)|dz < \infty, \quad 0 \leq s \leq q + \gamma_x + 2(p + \gamma_f).$$

Moreover, we suppose that the integrals

$$\int z^j K(z) dz = 0, \quad j = \overline{1, p+q}.$$

Assumption (h): Let $(h_n)_{n \geq 1}$ be a sequence of monotonically decreasing to zero positive numbers such that $nh_n \rightarrow \infty$ as $n \rightarrow \infty$ and $h_1 \leq 1$.

It should be noted that, according to Assumption (K), the allowed kernels can be of order higher than two, i.e. not necessarily nonnegative everywhere. For example, the infinite-order flat-top kernels of Politis (2001) are allowed, including the infinitely differentiable flat-top kernel, see McMurry and Politis (2004).

3. ESTIMATION OF THE REGRESSION FUNCTION WITH GUARANTEED ACCURACY

In this section we consider sequential estimators of the regression function f at a point x^0 with a given variance, calculated at a special stopping time. These estimators are constructed on the basis of Nadaraya-Watson kernel estimators. Bandwidth giving an optimal convergence rate of estimators (which coincides with the rate of Nadaraya-Watson estimators constructed by independent observations) is found.

3.1. Sequential Estimation Procedure

According to sequential approach we introduce the following definition.

Definition 3.1. Let $(H_n)_{n \geq 1}$ be an unboundedly monotonically increasing given sequence of positive numbers and $(h_n)_{n \geq 1}$ satisfies Assumption (h).

The sequential plans (τ_n, f_n^*) , $n \geq 1$ of estimation of the function $f = f(x^0)$ will be defined by the formulae

$$\begin{aligned} \tau_n &= \inf \{j \geq 1 : \sum_{i=1}^j K\left(\frac{x^0 - x_i}{h_n}\right) \geq H_n\}, \\ f_n^* &= \frac{1}{Q_n} \sum_{i=1}^{\tau_n} Y_i K\left(\frac{x^0 - x_i}{h_n}\right), \quad Q_n = \sum_{i=1}^{\tau_n} K\left(\frac{x^0 - x_i}{h_n}\right), \end{aligned} \quad (3.1)$$

where τ_n is the duration of estimation, and f_n^* is the estimator of f with given accuracy in the mean square sense.

We have introduced in Definition 3.1 the sequence of sequential estimation plans. At the same time Theorem 3.1 below gives non-asymptotic properties of these plans for every $n \geq 1$, in particular, upper bound for the MSE of the estimator f_n^* . It gives a possibility to investigate asymptotic properties of defined plans as $n \rightarrow \infty$ as well (see Section 3.2).

To formulate this theorem we need the following notation.

Assume $(l_n)_{n \geq 1}$ to be an arbitrary non-decreasing sequence of positive numbers.

Define the function $\gamma_{x,\delta} = 0$ if processes X and Δ are independent and 1 otherwise; $\gamma_\delta = 0$ if $\varphi(k) = 0$, $k > 0$ and 1 otherwise; the numbers

$$\begin{aligned}\varphi_n &= \varphi(l_n), \quad K_\alpha = \int |z^\alpha K(z)| dz, \quad \alpha \geq 0, \quad \kappa = (q!)^{-1} L_x K_{q+\gamma_x}, \\ C_1 &= \frac{1}{q!} \sum_{j=1}^p \frac{1}{j!} C_{f,j} K_{q+j+\gamma_x} \chi(p > 0), \quad C_2 = \frac{1}{p!} C_{x,0} K_{p+\gamma_f}, \quad C_3 = C_{x,0} \int K^2(y) dy, \\ C_4 &= 4\bar{K}(p+2) \sum_{j=0}^{q+1} \sum_{k=0}^{p+1} \frac{1}{j!(k!)^2} C_{x,j} C_{f,k}^2 K_{\beta(j,k)}, \quad C_{x,q+1} = \frac{1}{q!} L_x, \quad C_{f,p+1} = \frac{1}{p!} L_f, \\ \beta(j,k) &= j + 2k + (\gamma_x - 1)\chi(j = q+1) + 2(\gamma_f - 1)\chi(k = p+1),\end{aligned}$$

as well as

$$\begin{aligned}n_0 &= \inf\{n \geq 1 : c_x^0 - \kappa h_n^{q+\gamma_x} > 0\}, \\ t_n &= \frac{(C_{x,0} + \kappa h_n^{q+\gamma_x}) \cdot \sqrt{C_3(H_n + \bar{K})}}{(c_x^0 - \kappa h_n^{q+\gamma_x})^{5/2} h_n} \\ &+ \frac{\sqrt{[C_3(C_{x,0} + \kappa h_n^{q+\gamma_x})^2 \cdot (c_x^0 - \kappa h_n^{q+\gamma_x})^{-3} + 2\bar{K}^2(c_x^0 - \kappa h_n^{q+\gamma_x})^{-1}](H_n + \bar{K}) + H_n^2}}{(c_x^0 - \kappa h_n^{q+\gamma_x}) h_n}, \\ w_n &= \sqrt{\varphi(0)\bar{K}^2 \gamma_\delta \frac{l_n}{H_n^2} (1 + 3l_n) + C_3 \varphi(0) (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} \frac{1}{H_n} \left(1 + \frac{\bar{K}}{H_n}\right)} \\ &+ 2\gamma_\delta \varphi(0) K_0 \bar{K} (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} \frac{l_n}{H_n} \left(1 + \frac{\bar{K}}{H_n}\right) + 2\gamma_\delta K_0 \bar{K} \varphi_n \frac{h_n t_n^2}{H_n^2}, \\ b_n &= (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} [C_1 h_n^{q+1+\gamma_x} + C_2 h_n^{p+\gamma_f}] \left(1 + \frac{\bar{K}}{H_n}\right) + w_n\end{aligned}$$

and

$$\begin{aligned}V_n &= 2 \left\{ (2C_3 C_{f,0}^2 + (1 + \gamma_{x,\delta}) C_4) (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} \frac{1}{H_n} \left(1 + \frac{\bar{K}}{H_n}\right) \right. \\ &\quad \left. + (1 + \gamma_{x,\delta}) w_n^2 + \frac{2(C_1 h_n^{q+1+\gamma_x} + C_2 h_n^{p+\gamma_f})^2 h_n^2 t_n^2}{H_n^2} \right\}.\end{aligned}$$

Theorem 3.1. Assume model (1.1) and let the Assumptions (f), (K), (X), (h) and (Δ) be fulfilled and $n \geq n_0$. Then the sequential estimation plans (τ_n, f_n^*) are closed, i.e. $\tau_n < \infty$ a.s. and have the following non-asymptotic properties:

1. for the expected duration of observations

a)

$$(C_{x,0} + \kappa h_n^{q+\gamma_x})^{-1} h_n^{-1} H_n \leq E\tau_n \leq (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} h_n^{-1} (H_n + \bar{K}), \quad (3.2)$$

b)

$$E\tau_n^2 \leq t_n^2; \quad (3.3)$$

2. for the bias

$$|Ef_n^* - f| \leq b_n;$$

3. for the MSE

$$E(f_n^* - f)^2 \leq V_n.$$

Proof of Theorem 3.1 is given in the Appendix.

Remark 3.1. From Definition 3.1 it follows that the presented sequential estimators coincide with the usual Nadaraya–Watson estimators calculated at a special stopping time. Therefore, at least in the case of independent inputs of the model, these estimators have the same asymptotic properties as Nadaraya–Watson estimators. However, sequential estimators have the above exact, non-asymptotic properties, that may be important for practitioners.

3.2. Bandwidth Choice for Sequential Estimators

From Theorem 3.1 follows, that it is natural to define $H_n = nh_n$. In this case and under the condition on the bandwidth $nh_n \rightarrow \infty$ as $n \rightarrow \infty$ from Assumption (h) the expected duration of observations is proportional to n , i.e.,

$$C_{x,0}^{-1}n - \frac{C_{x,0}^{-1}\kappa h_n^{q+\gamma_x}n}{C_{x,0} + \kappa h_n^{q+\gamma_x}} \leq E\tau_n \leq (c_x^0)^{-1}n + \frac{(c_x^0)^{-1}\kappa h_n^{q+\gamma_x}n + \bar{K}h_n^{-1}}{c_x^0 - \kappa h_n^{q+\gamma_x}}, \quad n \geq n_0,$$

as well as

$$E\tau_n^2 \leq (c_x^0)^{-2}n^2 + o(n^2) \quad \text{as } n \rightarrow \infty$$

and

$$|Ef_n^* - f| = O\left(h_n^{q+1+\gamma_x} + h_n^{p+\gamma_f} + \sqrt{\varphi_n h_n^{-1}} + \frac{1 + \gamma_\delta \sqrt{l_n}}{\sqrt{nh_n}}\right),$$

$$E(f_n^* - f)^2 = O\left(h_n^{2(q+1+\gamma_x)} + h_n^{2(p+\gamma_f)} + \varphi_n h_n^{-1} + \frac{1 + \gamma_\delta l_n}{nh_n}\right) \quad \text{as } n \rightarrow \infty.$$

From these formulae it follows, that for the asymptotic unbiasedness and L_2 -convergency of f_n^* we have to use the sequences (l_n) and (h_n) satisfying the condition

$$\varphi_n h_n^{-1} + \frac{l_n}{nh_n} = o(1) \quad \text{as } n \rightarrow \infty. \quad (3.4)$$

Remark 3.2. If the regressors x_i form a sequence of i.i.d.r.v's, then $c_x^0 = C_{x,0}$. Moreover if the number $c_x^0 = f_i(x^0)$ is known, we can put $H_n = c_x^0 nh_n$. In this case

$$n - \frac{\kappa h_n^{q+\gamma_x}n}{c_x^0 + \kappa h_n^{q+\gamma_x}} \leq E\tau_n \leq n + \frac{\kappa h_n^{q+\gamma_x}n + \bar{K}h_n^{-1}}{c_x^0 - \kappa h_n^{q+\gamma_x}}$$

and

$$\overline{\lim}_{n \rightarrow \infty} [(h_n^{q+\gamma_x}n)^{-1} \wedge h_n] |E\tau_n - n| < \infty,$$

where $a \wedge b = \min\{a, b\}$, as well as

$$E\tau_n^2 \leq n^2 + o(n^2) \quad \text{as } n \rightarrow \infty.$$

It should be noted that in the case of i.i.d. noises (δ_i) the MSE of the constructed sequential estimator has similar optimal decreasing rate

$$E(f_n^* - f)^2 = O\left(h_n^{2(q+1+\gamma_x)} + h_n^{2(p+\gamma_f)} + \frac{1}{nh_n}\right) \quad \text{as } n \rightarrow \infty$$

as in the case of i.i.d. regressors (x_i) .

In particular, for the case $p = 0$, $\gamma_f = 1$ (see, for comparison, Györfi et al. (2002) among others),

$$E(f_n^* - f)^2 = O\left(h_n^2 + \frac{1}{nh_n}\right) \quad \text{as } n \rightarrow \infty.$$

4. TRUNCATED SEQUENTIAL ESTIMATION OF THE REGRESSION FUNCTION f

We shall consider in this section the problem of estimation of the regression function f with a known mean square accuracy based on observations of the process (Y_i, x_i) for $i = 1, \dots, N$ on the time interval $[1, N]$ for a fixed time N . These estimators are constructed on the basis of sequential estimators considered in the previous section.

Such possibility gives the so-called truncated sequential estimation method, developed in Konev and Pergamenshchikov (1990a)-Konev and Pergamenshchikov (1992), as well as in Fourdrinier et al. (2009) for parameter estimation problems in discrete and continuous time dynamic models.

Bandwidth giving an optimal convergence rate of estimators is found.

4.1. Truncated Sequential Estimation Procedure

Definition 4.1. Let H and h be positive numbers.

The truncated sequential plans $(\tau_N(h, H), f_N^*(h, H))$, $N \geq 1$ of estimation of the function $f = f(x^0)$ will be defined by the formulae

$$\tau_N(h, H) = \inf\{j \in [1, N] : \sum_{i=1}^j K\left(\frac{x^0 - x_i}{h}\right) \geq H\}$$

with the provision that $\inf\{\emptyset\} = N$,

$$f_N^*(h, H) = \hat{f}_N(h, H) \cdot \chi\left(\sum_{i=1}^{\tau_N(h, H)} K\left(\frac{x^0 - x_i}{h}\right) \geq H\right) + \frac{C_{f,0}}{2} \cdot \chi\left(\sum_{i=1}^{\tau_N(h, H)} K\left(\frac{x^0 - x_i}{h}\right) < H\right), \quad (4.1)$$

$$\hat{f}_N(h, H) = \frac{1}{Q_N(h, H)} \sum_{i=1}^{\tau_N(h, H)} Y_i K\left(\frac{x^0 - x_i}{h}\right), \quad Q_N(h, H) = \sum_{i=1}^{\tau_N(h, H)} K\left(\frac{x^0 - x_i}{h}\right),$$

where $\tau_N(h, H)$ is the duration of estimation, which is bounded by construction ($\tau_N(h, H) \leq N$) and $f_N^*(h, H)$ is the estimator of f . The notation $\chi(A)$ means the indicator function of set A .

The parameters H and h will be defined in a special way as functions of N for optimization of a convergence rate of the estimator $f_N^*(h, H)$ as $N \rightarrow \infty$ in the mean square sense.

Assume $l(H)$ to be an arbitrary non-decreasing function of positive numbers.

Define

$$\begin{aligned} V_N(h, H) = & 2 \left\{ (2C_3 C_{f,0}^2 + (1 + \gamma_{x,\delta}) C_4) \frac{hN}{H^2} + (1 + \gamma_{x,\delta}) \left[\varphi(0) \bar{K}^2 \gamma_\delta \frac{l(H)}{H^2} (1 + 3l(H)) \right. \right. \\ & \left. \left. + C_3 \varphi(0) \frac{hN}{H^2} + 2\gamma_\delta \varphi(0) K_0 \frac{l(H)hN}{H^2} + 2\gamma_\delta K_0 \bar{K} \varphi(l(H)) \frac{hN^2}{H^2} \right] \right. \\ & \left. + \frac{2(C_1 h^{q+1+\gamma_x} + C_2 h^{p+\gamma_f})^2 h^2 N^2}{H^2} \right\} + \frac{C_3 C_{f,0}^2 N h}{4[Nh(c_x^0 - \kappa h^{q+\gamma_x}) - H]^2}. \end{aligned}$$

From the proof of the following Theorem 4.1 it follows that the number H satisfies the condition

$$Nh(c_x^0 - \kappa h^{q+\gamma_x}) - H > 0. \quad (4.2)$$

Thus it is natural to put

$$H := H_N = Nh_N C_\alpha^{-1}, \quad (4.3)$$

where

$$C_\alpha = (c_x^0 - \kappa h_{n_0}^{q+\gamma_x} - \alpha^{-1/2})^{-1}, \quad \alpha > (c_x^0 - \kappa h_{n_0}^{q+\gamma_x})^{-2}$$

and (h_N) is a sequence satisfying Assumption (h). It should be noted that, according to the assumptions of the theorem below the number C_α and the sequence (H_N) are assumed known.

Then, denoting $l_N = l(H_N)$ and $\varphi_N = \varphi(l_N)$, we have

$$\begin{aligned} V_N^0 := V_N(h_N, H_N) = & 2 \left\{ [(2C_3C_{f,0}^2 + (1 + \gamma_{x,\delta})C_4)C_\alpha^2 + \frac{C_3C_{f,0}^2\alpha}{8} + (1 + \gamma_{x,\delta}) \right. \\ & \times (C_3C_\alpha^2\varphi(0) + 2\gamma_\delta\varphi(0)K_0C_\alpha^2l_N)] \frac{1}{Nh_N} + (1 + \gamma_{x,\delta})[\varphi(0)\bar{K}^2C_\alpha^2\gamma_\delta \frac{l_N}{(Nh_N)^2}(1 + 3l_N) \\ & \left. + 2\gamma_\delta K_0\bar{K}C_\alpha^2\varphi_N h_N^{-1}] + 2C_\alpha^2(C_1h_N^{q+1+\gamma_x} + C_2h_N^{p+\gamma_f})^2 \right\}. \end{aligned}$$

The main result of this section is the following

Theorem 4.1. Assume model (1.1) and let the Assumptions (f), (K), (X) and (Δ) be fulfilled, where the number $C_{f,0}$ in Assumption (f) is supposed to be known. Then:

1) for every positive numbers h, H satisfying the condition (4.2) the truncated sequential estimator $f_N^*(h, H)$ has the MSE:

$$E(f_N^*(h, H) - f)^2 \leq V_N(h, H);$$

2) for $H = (H_N)$ defined in (4.3) and (h_N) satisfying (4.2) it holds

$$E(f_N^*(h_N, H_N) - f)^2 \leq V_N^0;$$

3) assume there exists a number $r > 2$, such that the bandwidth $h = (h_N)$ from Assumption (h) satisfies the additional condition

$$\sum_{N \geq 1} \frac{1}{h_N^{r-1} N^{r/2}} < \infty \quad (4.4)$$

and $H = (H_N)$ is defined in (4.3). Then

$$\lim_{N \rightarrow \infty} \frac{\tau_N(h_N, H_N)}{N} < 1 \quad \text{a.s.}$$

and the estimator $f_N^*(h_N, H_N)$ is non-degenerate as $N \rightarrow \infty$ in the following sense:

$$P(f_N^*(h_N, H_N) = \hat{f}_N(h_N, H_N)) = 1 \quad \text{for } N \text{ large enough.}$$

Proof of Theorem 4.1 is given in the Appendix.

Remark 4.1. From the third assertion of Theorem 4.1 it follows, that similarly to sequential estimators, the truncated sequential estimators have the asymptotic properties of Nadaraya–Watson estimators (2.1). However, in distinction from the sequential estimators considered in Section 3.1, the truncated estimators have known variance for fixed (non-random) sample sizes.

Remark 4.2. If the number $C_{f,0}$ in Assumption (f) is unknown, then the Definition 2.2 of sequential estimator is modified as follows:

$$f_N^*(h, H) = \hat{f}_N(h, H) \cdot \chi \left(\sum_{i=1}^N K \left(\frac{x^0 - x_i}{h} \right) \geq H \right).$$

All assertions of Theorem 4.1 will be fulfilled with slightly changed (but having similar structure) functions $V_N(h, H)$ and V_N^0 . More precisely, the number 4 in the denominator of the last summand in the definition of $V_N(h, H)$ should be omitted and the number 8 in the denominator of the third summand in the right hand side in the definition of V_N^0 should be changed to the number 2.

4.2. Bandwidth Choice for Truncated Sequential Estimators

From Theorem 4.1 it follows, that by the definition of V_N^0 and assertion 2) of Theorem 4.1, the truncated sequential estimator (4.1) has similar to the sequential estimator (3.1) rate of convergency (as $N \rightarrow \infty$).

In the case of i.i.d. noises (δ_i) the MSE of the constructed sequential estimator has optimal decreasing rate

$$E(f_N^*(h_N, H_N) - f)^2 = O\left(h_N^{2\varrho} + \frac{1}{Nh_N}\right) \quad \text{as } N \rightarrow \infty,$$

$$\varrho = (q + 1 + \gamma_x) \wedge (p + \gamma_f).$$

Then the bandwidth h with the optimal rate is proportional to

$$h_N \sim N^{-\frac{1}{(2\varrho+1)}}$$

and the condition (4.4) is fulfilled for $\varrho > 1/2$ and

$$r > \frac{4\varrho}{2\varrho - 1}.$$

In particular, for the case $p = 0$, $\gamma_f = 1$,

$$E(f_N^*(h_N, H_N) - f)^2 = O\left(h_N^2 + \frac{1}{Nh_N}\right) \quad \text{and } r > 4.$$

Remark 4.3. Theorems 3.1 and 4.1 give known upper bounds for the bias and the MSE of presented estimators as well as for the mean of observation time in Theorem 3.1 if the parameters of classes of functions, introduced in Assumptions (f) and (X) are known ($C_{f,k}, p, \gamma_f$, etc). In this case 'optimal' bandwidth can be found from minimization of the upper bounds for the MSE's.

At the same time the assertions of Theorems 3.1 and 4.1 are fulfilled even if some of these parameters are unknown. Estimators with such properties can be successfully used in various adaptive procedures as pilot estimators (see, e.g., Dobrovodov et al. (2012), Vasiliev (1997)-Vasiliev and Koshkin (1998)).

5. EXAMPLES

In this section we consider examples of dependence types of observed regressors and model noises satisfying Assumptions (X) and (Δ).

5.1. Case of Independent Regressors X and Δ

Consider examples of inputs of the model in this case.

5.1.1. Example of Regressors X

Assume that regressors (x_i) satisfy the following equation

$$x_i = \Psi(x_{i-1}, \dots, x_{i-r}) + \varepsilon_i, \quad i \geq 1, \quad (5.1)$$

where $\Psi(\cdot)$ is a bounded function $|\Psi(y)| \leq \bar{\Psi} < \infty$, $y \in \mathcal{R}^r$, $r \geq 1$.

Assume that the input noises ε_i in the model (5.1) are i.i.d. with the density function $f_\varepsilon(\cdot)$ and independent from the initial vector $(x_0, x_{-1}, \dots, x_{1-r})$. In this case $f_{i-1}(t) = f_\varepsilon(t - \Psi(x_{i-1}, \dots, x_{i-r}) | \mathcal{F}_{i-1})$.

Thus Assumption (X) holds true if the density function $f_\varepsilon(\cdot)$ is q -th differentiable, $q \geq 0$ and for some positive numbers c_x^0 , $C_{x,0}$, L_x and $\gamma_x \in (0, 1]$ the following relations

$$\inf_{|t-x^0| \leq \bar{\Psi}} f_\varepsilon(t) \geq c_x^0,$$

$$\sup_{|t-x^0| \leq \bar{\Psi}} |f_\varepsilon^{(i)}(t)| \leq C_{x,i}, \quad i = \overline{0, q}$$

and for every $x, y \in \mathbb{R}^1$,

$$|f_\varepsilon^{(q)}(x) - f_\varepsilon^{(q)}(y)| \leq L_x |x - y|^{\gamma_x}$$

are fulfilled.

5.1.2. Example of Noises Δ

Assume that noises (δ_i) satisfy the following autoregressive equation

$$\delta_i = \lambda \delta_{i-1} + \eta_i, \quad i \geq 1, \quad (5.2)$$

which is supposed to be stable $|\lambda| < 1$, and η_i are i.i.d., $E\eta_i = 0$, $E\eta_i^2 \leq \sigma_\eta^2$, $i \geq 0$; $\eta_0 = \delta_0$. Then

$$\varphi(k) = \frac{|\lambda|^k \sigma_\eta^2}{1 - \lambda^2}.$$

In this case we can put (after optimization of the upper bound of the MSE)

$$l_n \sim (1 + \mu) \log_{|\lambda|} \frac{1}{n}, \quad \mu > 0$$

and the summands

$$\frac{l_n}{nh_n} \sim \frac{\log n}{nh_n}, \quad \varphi_n h_n^{-1} \sim \frac{1}{n^{1+\mu} h_n} = o\left(\frac{1}{nh_n}\right).$$

Then the condition (3.4) (which is necessary for truncated estimators as well) is fulfilled if

$$\frac{\log n}{nh_n} = o(1) \quad \text{as } n \rightarrow \infty.$$

It is clear, that this example can be easily generalized for the stable autoregressive process (5.2) of an arbitrary order.

5.2. Case of Dependent Inputs X and Δ

Consider the estimation problem in the nonlinear autoregressive model

$$Y_i = f(Y_{i-1}) + \delta_i, \quad i \geq 1,$$

where δ_i , $i \geq 1$ form the sequence of zero mean i.i.d.r.v's with a density function $f_\delta(\cdot)$ and finite variance $E\delta_i^2 = \sigma^2$, $i \geq 1$; Y_0 is a zero mean random variable with finite variance and independent from δ_i , $i \geq 1$.

Then $x_n = Y_{n-1}$, $n \geq 0$ and $\varphi(k) = \sigma^2 \chi(k = 0)$, $k \geq 0$ and in this case $f_i(t) = f_\delta(t - f(Y_{i-1})) | \mathcal{F}_{i-1}^Y$, $\mathcal{F}_i^Y = \sigma\{Y_0, \delta_1, \dots, \delta_i\}$.

Thus Assumption (X) holds true if the function $f(\cdot)$ is uniformly bounded $\sup_{t \in \mathbb{R}^1} f(t) \leq C_f$, the density function $f_\delta(\cdot)$ is q -th differentiable, $q \geq 0$ and for some positive numbers c_x^0 , $C_{x,0}$, L_x and $\gamma_x \in (0, 1]$ the following relations

$$\inf_{|t-x^0| \leq C_f} f_\delta(t) \geq c_x^0,$$

$$\sup_{|t-x^0| \leq C_f} f_\delta^{(i)}(t) \leq C_{x,i}, \quad i = \overline{0, q}$$

and for every $x, y \in \mathbb{R}^1$,

$$|f_\delta^{(q)}(x) - f_\delta^{(q)}(y)| \leq L_x |x - y|^{\gamma_x}$$

are fulfilled.

6. CONCLUSION

The sequential approach for the problem of estimation of a regression function from dependent observations is developed. It is supposed, that the function to be estimated belongs to a Hölder class and input processes of the model can be unbounded with a positive probability.

Two types estimators are presented. Sequential estimators give the possibility to get estimators with an arbitrary mean square accuracy by finite stopping time. Truncated sequential estimators have known variance and constructed by a sample of finite (fixed) size.

Both estimation procedures work under general dependency types of model inputs. Asymptotic rate of convergence of both estimators coincides with the optimal rate of Nadaraya–Watson estimators constructed from independent observations. At that we consider the mean $E\tau_n$ having the rate (3.2) as a duration of observations in sequential approach for comparison with Nadaraya–Watson estimators calculated by the sample of the size n .

Presented estimators can be used directly and as pilot estimators in various statistical problems.

In this paper we have assumed that the function to be estimated has a scalar argument. In most applications, in particular to dynamic systems, the regressors will have a higher dimension. The extension to this case is immediate.

7. APPENDIX

7.1. Proof of Theorem 3.1

First we note that by the Definition 3.1 for the closeness of stopping times τ_n , $n \geq 1$ it is enough to verify for every $n \geq n_0$ the following equalities

$$\sum_{i=1}^{\infty} K \left(\frac{x^0 - x_i}{h_n} \right) = +\infty \quad \text{a.s.} \quad (7.1)$$

To this end we consider for every $n \geq n_0$ and $N \geq 1$ the following representation

$$\begin{aligned} \frac{1}{Nh_n} \sum_{i=1}^N K \left(\frac{x^0 - x_i}{h_n} \right) &= \frac{1}{Nh_n} \sum_{i=1}^N E \left[K \left(\frac{x^0 - x_i}{h_n} \right) \middle| \mathcal{F}_{i-1} \right] \\ &+ \frac{1}{Nh_n} \sum_{i=1}^N \left(K \left(\frac{x^0 - x_i}{h_n} \right) - E \left[K \left(\frac{x^0 - x_i}{h_n} \right) \middle| \mathcal{F}_{i-1} \right] \right). \end{aligned}$$

Using the Assumptions (X) and (K), the first summand in the right hand side of this formula can be estimated from below

$$\begin{aligned}
& \frac{1}{Nh_n} \sum_{i=1}^N E[K \left(\frac{x^0 - x_i}{h_n} \right) | \mathcal{F}_{i-1}] = \frac{1}{N} \sum_{i=1}^N \int K(z) f_{i-1}(x^0 - h_n z) dz \\
&= \frac{1}{N} \sum_{i=1}^N f_{i-1}(x^0) + \frac{1}{N} \sum_{i=1}^N \int K(z) [f_{i-1}(x^0 - h_n z) - f_{i-1}(x^0)] dz \\
&= \frac{1}{N} \sum_{i=1}^N f_{i-1}(x^0) + \sum_{j=1}^q \frac{(-1)^j h_n^j}{j!} \frac{1}{N} \sum_{i=1}^N f_{i-1}^{(j)}(x^0) \int z^j K(z) dz \\
&+ \frac{(-1)^q h_n^q}{q!} \frac{1}{N} \sum_{i=1}^N \int z^q K(z) [f_{i-1}^{(q)}(x^0 - \theta_{q,n,z} h_n z) - f_{i-1}^{(q)}(x^0)] dz \\
&\geq c_x^0 - \kappa h_n^{q+\gamma_x} \geq c_x^0 - \kappa h_{n_0}^{q+\gamma_x} > 0,
\end{aligned}$$

where the number $\theta_{q,n,z} \in [0, 1]$.

As regards to the second summand we note, that for every fixed $n \geq n_0$ according to Assumption (K) the processes $(M_{n,N})_{N \geq 1}$,

$$M_{n,N} = \sum_{i=1}^N \frac{1}{i} \left\{ K \left(\frac{x^0 - x_i}{h_n} \right) - E[K \left(\frac{x^0 - x_i}{h_n} \right) | \mathcal{F}_{i-1}] \right\}$$

form square integrable martingales with uniformly bounded variances:

$$\sup_{N \geq 1} EM_{n,N}^2 < \infty, \quad n \geq 1.$$

Thus, from the Doob theorem for square integrable martingales, see Liptser and Shiryaev (1988) and the Kronecker lemma (see, e.g., Loeve (1963)), the second summand goes to zero a.s.

Therefore the relations (7.1) and the finiteness of the stopping times τ_n are proven.

For the proof of the first assertion of Theorem 3.1 we note that the stopping times τ_n , $n \geq 1$ are $\{\mathcal{F}_{i-1}\}$ -adaptive, i.e. $\{i \leq \tau_n\} \in \mathcal{F}_{i-1}$, $i > 1$, $n \geq 1$. Thus, by the definition of τ_n and using a well-known technique of sequential analysis (see, e.g., Borisov and Konev (1977); Dobrovidov et al. (2012); Konev (1985); Kchler and Vasiliev (2010); Liptser and Shiryaev (1978); Wald (1947, 1950) among others) and Assumption (X), we have

$$\begin{aligned}
H_n &> E \sum_{i=1}^{\tau_n-1} K \left(\frac{x^0 - x_i}{h_n} \right) = E \sum_{i=1}^{\tau_n} K \left(\frac{x^0 - x_i}{h_n} \right) - EK \left(\frac{x^0 - x_{\tau_n}}{h_n} \right) \\
&\geq E \sum_{i=1}^{\tau_n} E[K \left(\frac{x^0 - x_i}{h_n} \right) | \mathcal{F}_{i-1}] - \bar{K} = h_n E \sum_{i=1}^{\tau_n} \int K(z) f_{i-1}(x^0 - h_n z) dz - \bar{K} \\
&= h_n E \sum_{i=1}^{\tau_n} f_{i-1}(x^0) + h_n E \sum_{i=1}^{\tau_n} \int K(z) (f_{i-1}(x^0 - h_n z) - f_{i-1}(x^0)) dz - \bar{K} \\
&= h_n E \sum_{i=1}^{\tau_n} f_{i-1}(x^0) + \sum_{j=1}^q \frac{(-1)^j h_n^{j+1}}{j!} E \sum_{i=1}^{\tau_n} f_{i-1}^{(j)}(x^0) \int z^j K(z) dz
\end{aligned}$$

$$\begin{aligned}
& + \frac{(-1)^q}{q!} h_n^{q+1} E \sum_{i=1}^{\tau_n} \int z^m K(z) [f_{i-1}^{(q)}(x^0 - \theta_{q,n,z} h_n z) - f_{i-1}^{(q)}(x^0)] dz \\
& \geq c_x^0 h_n E \tau_n - \kappa h_n^{q+1+\gamma_x} E \tau_n - \bar{K}.
\end{aligned}$$

Then

$$E \tau_n \leq (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} h_n^{-1} (H_n + \bar{K}), \quad n \geq n_0.$$

On the other hand

$$\begin{aligned}
H_n & \leq E \sum_{i=1}^{\tau_n} K \left(\frac{x^0 - x_i}{h_n} \right) = h_n E \sum_{i=1}^{\tau_n} \int K(z) f_{i-1}(x^0 - h_n z) dz \\
& = h_n E \sum_{i=1}^{\tau_n} f_{i-1}(x^0) + h_n E \sum_{i=1}^{\tau_n} \int K(z) (f_{i-1}(x^0 - h_n z) - f_{i-1}(x^0)) dz \\
& \leq h_n (C_{x,0} + \kappa h_n^{q+\gamma_x}) E \tau_n
\end{aligned}$$

and as follows

$$E \tau_n \geq (C_{x,0} + \kappa h_n^{q+\gamma_x})^{-1} h_n^{-1} H_n, \quad n \geq 1.$$

Similarly, we have

$$\begin{aligned}
H_n^2 & > E \left[\sum_{i=1}^{\tau_n-1} K \left(\frac{x^0 - x_i}{h_n} \right) \right]^2 \geq E \left[\sum_{i=1}^{\tau_n} K \left(\frac{x^0 - x_i}{h_n} \right) \right]^2 \\
& - 2\bar{K} E \sum_{i=1}^{\tau_n} K \left(\frac{x^0 - x_i}{h_n} \right) \geq E \left\{ \sum_{i=1}^{\tau_n} E \left[K \left(\frac{x^0 - x_i}{h_n} \right) \mid \mathcal{F}_{i-1} \right] \right\}^2 \\
& - 2(C_{x,0} + \kappa h_n^{q+\gamma_x}) h_n \sqrt{E \tau_n^2} \cdot \sqrt{E \sum_{i=1}^{\tau_n} \left(K \left(\frac{x^0 - x_i}{h_n} \right) - E \left[K \left(\frac{x^0 - x_i}{h_n} \right) \mid \mathcal{F}_{i-1} \right] \right)^2} \\
& - 2\bar{K} h_n \sum_{i=1}^{\tau_n} \int K(y) f_{i-1}(x^0 - h_n y) dy \geq (c_x^0 - \kappa h_n^{q+\gamma_x})^2 h_n^2 E \tau_n^2 - 2(C_{x,0} + \kappa h_n^{q+\gamma_x}) h_n \sqrt{E \tau_n^2} \\
& \times \sqrt{h_n E \sum_{i=1}^{\tau_n} \int [K(y) - E[K \left(\frac{x^0 - x_i}{h_n} \right) \mid \mathcal{F}_{i-1}]]^2 f_{i-1}(x^0 - h_n y) dy - 2\bar{K} K_0 C_{x,0} h_n E \tau_n} \\
& \geq (c_x^0 - \kappa h_n^{q+\gamma_x})^2 h_n^2 E \tau_n^2 - 2(C_{x,0} + \kappa h_n^{q+\gamma_x}) h_n \sqrt{E \tau_n^2} \cdot \sqrt{C_3 h_n E \tau_n} \\
& - 2\bar{K} K_0 C_{x,0} h_n (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} h_n^{-1} (H_n + \bar{K}) \geq (c_x^0 - \kappa h_n^{q+\gamma_x})^2 h_n^2 \cdot E \tau_n^2 \\
& - 2(C_{x,0} + \kappa h_n^{q+\gamma_x}) h_n \cdot \sqrt{C_3 (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} (H_n + \bar{K})} \cdot \sqrt{E \tau_n^2} \\
& - 2\bar{K} K_0 C_{x,0} (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} (H_n + \bar{K}).
\end{aligned}$$

Thus we have

$$E \tau_n^2 \leq \left\{ \frac{(C_{x,0} + \kappa h_n^{q+\gamma_x}) \cdot \sqrt{C_3 (H_n + \bar{K})}}{(c_x^0 - \kappa h_n^{q+\gamma_x})^{5/2} h_n} \right\}$$

$$+ \frac{\sqrt{[C_3(C_{x,0} + \kappa h_n^{q+\gamma_x})^2 \cdot (c_x^0 - \kappa h_n^{q+\gamma_x})^{-3} + 2\bar{K}^2(c_x^0 - \kappa h_n^{q+\gamma_x})^{-1}](H_n + \bar{K}) + H_n^2}}{(c_x^0 - \kappa h_n^{q+\gamma_x})h_n}} \Bigg\}^2.$$

Then the first assertion of the theorem follows from the definition of the function t_n .

Further, from Definition 3.1 and (1.1) we find the formula for the deviation of the sequential estimator f_n^* :

$$f_n^* - f = I_n + J_n, \quad (7.2)$$

where

$$I_n = \frac{1}{Q_n} \sum_{i=1}^{\tau_n} (f(x_i) - f(x^0)) K\left(\frac{x^0 - x_i}{h_n}\right), \quad J_n = \frac{1}{Q_n} \sum_{i=1}^{\tau_n} \delta_i K\left(\frac{x^0 - x_i}{h_n}\right).$$

Consider the bias of the estimator f_n^* using the representation (7.2) for its deviation and a technique of conditional mathematical expectations.

$$\begin{aligned} EI_n &= E \frac{1}{Q_n} \sum_{i=1}^{\tau_n} (f(x_i) - f(x^0)) K\left(\frac{x^0 - x_i}{h_n}\right) \\ &= E \frac{h_n}{Q_n} \sum_{i=1}^{\tau_n} \int K(z) (f(x^0 - h_n z) - f(x^0)) f_{i-1}(x^0 - h_n z) dz. \end{aligned}$$

By the Tailor expansion for the functions $f(x^0 - h_n z) - f(x^0)$ and $f_{i-1}(x^0 - h_n z)$ at a point x^0 up to the orders p and q respectively (Assumptions (f) and (X)) and according to Assumption (K) we have

$$\begin{aligned} &\left| \int K(z) (f(x^0 - h_n z) - f(x^0)) f_{i-1}(x^0 - h_n z) dz \right| = \left| \frac{(-1)^q}{q!} h_n^q \right. \\ &\times \sum_{j=1}^p \frac{(-1)^j}{j!} h_n^j f^{(j)}(x^0) \int z^{q+j} K(z) (f_{i-1}^{(q)}(x^0 - \theta_{q,n,z} h_n z) - f_{i-1}^{(q)}(x^0)) dz \\ &+ \frac{(-1)^p}{p!} h_n^p \int z^p K(z) (f^{(p)}(x^0 - \tilde{\theta}_{q,n,z} h_n z) - f^{(p)}(x^0)) f_{i-1}(x^0 - h_n z) dz \left. \right| \\ &\leq C_1 h_n^{q+1+\gamma_x} + C_2 h_n^{p+\gamma_f}, \end{aligned} \quad (7.3)$$

where $\tilde{\theta}_{q,n,z} \in [0, 1]$.

Thus, using the first assertion of the theorem, we obtain

$$\begin{aligned} |EI_n| &\leq \frac{1}{H_n} [C_1 h_n^{q+2+\gamma_x} + C_2 h_n^{p+1+\gamma_f}] E \tau_n \\ &\leq \frac{1}{H_n} [C_1 h_n^{q+2+\gamma_x} + C_2 h_n^{p+1+\gamma_f}] (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} h_n^{-1} (H_n + \bar{K}) \\ &= (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} [C_1 h_n^{q+1+\gamma_x} + C_2 h_n^{p+\gamma_f}] \left(1 + \frac{\bar{K}}{H_n}\right). \end{aligned} \quad (7.4)$$

According to Assumptions (K) and (Δ) we have

$$E J_n^2 \cdot \chi_{\{\tau_n \leq l_n\}} \leq \gamma_\delta \frac{1}{H_n^2} \sum_{i=1}^{l_n} \sum_{j=1}^{l_n} E |K\left(\frac{x^0 - x_i}{h_n}\right) K\left(\frac{x^0 - x_j}{h_n}\right)| \cdot \varphi(|i - j|)$$

$$\leq \varphi(0)\overline{K}^2\gamma_\delta\frac{l_n}{H_n^2}(1+l_n). \quad (7.5)$$

Further, by Definition 3.1 we can estimate

$$\begin{aligned} EJ_n^2 \cdot \chi_{\{\tau_n > l_n\}} &\leq \frac{\varphi(0)}{H_n^2} E \sum_{i=1}^{\tau_n} K^2 \left(\frac{x^0 - x_i}{h_n} \right) \\ &\quad + \frac{2}{H_n^2} E \sum_{1 \leq i < j \leq \tau_n} \varphi(j-i) K \left(\frac{x^0 - x_i}{h_n} \right) K \left(\frac{x^0 - x_j}{h_n} \right) \cdot \chi_{\{\tau_n > l_n\}} \\ &\leq \frac{\varphi(0)}{H_n^2} E \sum_{i=1}^{\tau_n} E[K^2 \left(\frac{x^0 - x_i}{h_n} \right) | \mathcal{F}_{i-1}^x] + \frac{2}{H_n^2} ES_n \leq \varphi(0)C_3 \frac{h_n E \tau_n}{H_n^2} + \frac{2}{H_n^2} ES_n, \end{aligned} \quad (7.6)$$

where

$$S_n = \sum_{j=2}^{\tau_n} K \left(\frac{x^0 - x_j}{h_n} \right) \sum_{i=1}^{j-1} K \left(\frac{x^0 - x_i}{h_n} \right) \varphi(j-i) \cdot \chi_{\{\tau_n > l_n\}}.$$

We have

$$\begin{aligned} S_n &= \sum_{j=2}^{l_n} K \left(\frac{x^0 - x_j}{h_n} \right) \sum_{i=1}^{j-1} K \left(\frac{x^0 - x_i}{h_n} \right) \varphi(j-i) \cdot \chi_{\{\tau_n > l_n\}} \\ &\quad + \sum_{j=l_n+1}^{\tau_n} K \left(\frac{x^0 - x_j}{h_n} \right) \sum_{i=1}^{j-l_n} K \left(\frac{x^0 - x_i}{h_n} \right) \varphi(j-i) \cdot \chi_{\{\tau_n > l_n\}} \\ &\quad + \sum_{j=l_n+1}^{\tau_n} K \left(\frac{x^0 - x_j}{h_n} \right) \sum_{i=j-l_n+1}^{j-1} K \left(\frac{x^0 - x_i}{h_n} \right) \varphi(j-i) \cdot \chi_{\{\tau_n > l_n\}} \\ &\leq \gamma_\delta \varphi(0) \left[\sum_{j=1}^{l_n} \left| K \left(\frac{x^0 - x_j}{h_n} \right) \right| \right]^2 + \gamma_\delta \varphi_n \sum_{j=l_n+1}^{\tau_n} \left| K \left(\frac{x^0 - x_j}{h_n} \right) \right| \sum_{i=1}^{j-l_n} \left| K \left(\frac{x^0 - x_i}{h_n} \right) \right| \cdot \chi_{\{\tau_n > l_n\}} \\ &\quad + \gamma_\delta \varphi(0) \sum_{j=l_n+1}^{\tau_n} \left| K \left(\frac{x^0 - x_j}{h_n} \right) \right| \sum_{i=j-l_n+1}^{j-1} \left| K \left(\frac{x^0 - x_i}{h_n} \right) \right| \cdot \chi_{\{\tau_n > l_n\}} \\ &\leq \gamma_\delta \varphi(0) l_n^2 \overline{K}^2 + \gamma_\delta K_0 h_n \varphi_n E \tau_n \sum_{j=1}^{\tau_n} \left| K \left(\frac{x^0 - x_j}{h_n} \right) \right| + \gamma_\delta \varphi(0) \overline{K} l_n \sum_{j=1}^{\tau_n} \left| K \left(\frac{x^0 - x_j}{h_n} \right) \right| \\ &\leq \gamma_\delta \varphi(0) l_n^2 \overline{K}^2 + \gamma_\delta K_0 \overline{K} h_n \varphi_n E \tau_n^2 + \gamma_\delta \varphi(0) K_0 \overline{K} l_n h_n E \tau_n. \end{aligned} \quad (7.7)$$

Then, by the definition of w_n and from (3.2), (3.3) and (7.5)-(7.7) it follows

$$\begin{aligned} EJ_n^2 &\leq \varphi(0)\overline{K}^2\gamma_\delta\frac{l_n}{H_n^2}(1+3l_n) + C_3\varphi(0)\frac{h_n E \tau_n}{H_n^2} \\ &\quad + 2\gamma_\delta\varphi(0)K_0\overline{K}\frac{l_n h_n E \tau_n}{H_n^2} + 2\gamma_\delta K_0\overline{K}\varphi_n\frac{h_n E \tau_n^2}{H_n^2} \leq w_n^2 \end{aligned} \quad (7.8)$$

and, by the definition of b_n as well as (7.4), (7.8) we have

$$|Ef_n^* - f| \leq |EI_n| + \sqrt{EJ_n^2} \leq b_n.$$

The second assertion of Theorem 3.1 is proved.

For the proof of the third assertion we define the function

$$\begin{aligned}\bar{f}_n^* &= \frac{1}{Q_n} \sum_{i=1}^{\tau_n} E \left[K \left(\frac{x^0 - x_i}{h_n} \right) f(x_i) | \mathcal{F}_{i-1} \right] \\ &= \frac{h_n}{Q_n} \sum_{i=1}^{\tau_n} \int K(z) f(x^0 - h_n z) f_{i-1}(x^0 - h_n z) dz.\end{aligned}$$

Now we estimate the second moment

$$\begin{aligned}E(f_n^* - \bar{f}_n^*)^2 &\leq (1 + \gamma_{x,\delta}) \left[E \frac{1}{Q_n^2} \left(\sum_{i=1}^{\tau_n} [f(x_i) K \left(\frac{x^0 - x_i}{h_n} \right) \right. \right. \\ &\quad \left. \left. - E \left[K \left(\frac{x^0 - x_i}{h_n} \right) f(x_i) | \mathcal{F}_{i-1} \right] \right)^2 + E J_n^2 \right] \\ &\leq (1 + \gamma_{x,\delta}) \left[\frac{1}{H_n^2} E \sum_{i=1}^{\tau_n} [f(x_i) K \left(\frac{x^0 - x_i}{h_n} \right) - E \left[K \left(\frac{x^0 - x_i}{h_n} \right) f(x_i) | \mathcal{F}_{i-1} \right]]^2 + E J_n^2 \right] \\ &= (1 + \gamma_{x,\delta}) \left[\frac{h_n}{H_n^2} E \sum_{i=1}^{\tau_n} \int [K(y) f(x^0 - h_n y) \right. \\ &\quad \left. - E \left[K \left(\frac{x^0 - x_i}{h_n} \right) f(x_i) | \mathcal{F}_{i-1} \right]]^2 f_{i-1}(x^0 - h_n y) dy + E J_n^2 \right] \\ &\leq (1 + \gamma_{x,\delta}) \frac{4h_n}{H_n^2} E \sum_{i=1}^{\tau_n} \int K^2(y) f^2(x^0 - h_n y) f_{i-1}(x^0 - h_n y) dy + w_n^2.\end{aligned}$$

Thus, using Tailor expansion for the functions $f(x^0 - h_n y)$, $f_{i-1}(x^0 - h_n y)$ at a point x^0 we estimate

$$\begin{aligned}f^2(x^0 - h_n y) &\leq (p+2) \sum_{k=0}^{p+1} \frac{1}{(k!)^2} C_{f,k}^2 y^{2k+2(\gamma_f-1)\chi(k=p+1)}, \\ |f_{i-1}(x^0 - h_n y)| &\leq \sum_{j=0}^{q+1} \frac{1}{j!} C_{x,j} |y|^{j+(\gamma_x-1)\chi(j=q+1)}\end{aligned}$$

and by the definition of C_4 we have

$$\begin{aligned}E(f_n^* - \bar{f}_n^*)^2 &\leq (1 + \gamma_{x,\delta}) \left[C_4 \frac{h_n}{H_n^2} E \tau_n + w_n^2 \right] \\ &\leq (1 + \gamma_{x,\delta}) \left[\frac{C_4}{H_n} (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} \left(1 + \frac{\bar{K}}{H_n} \right) + w_n^2 \right].\end{aligned}\tag{7.9}$$

Consider the difference

$$\bar{f}_n^* - f = \frac{h_n}{Q_n} \sum_{i=1}^{\tau_n} \int K(z) (f(x^0 - h_n z) - f(x^0)) f_{i-1}(x^0 - h_n z) dz$$

$$+f \frac{1}{Q_n} \sum_{i=1}^{\tau_n} \left\{ E \left[K \left(\frac{x^0 - x_i}{h_n} \right) | \mathcal{F}_{i-1} \right] - K \left(\frac{x^0 - x_i}{h_n} \right) \right\}.$$

Using (7.3), we have

$$\begin{aligned} E(\bar{f}_n^* - f)^2 &\leq 2E \left(\frac{h_n}{Q_n} \sum_{i=1}^{\tau_n} \int K(z) (f(x^0 - h_n z) - f(x^0)) f_{i-1}(x^0 - h_n z) dz \right)^2 \\ &\quad + 2E \left(f \frac{1}{Q_n} \sum_{i=1}^{\tau_n} \left\{ E \left[K \left(\frac{x^0 - x_i}{h_n} \right) | \mathcal{F}_{i-1} \right] - K \left(\frac{x^0 - x_i}{h_n} \right) \right\} \right)^2 \\ &\leq \frac{2h_n^2 (C_1 h_n^{q+1+\gamma_x} + C_2 h_n^{p+\gamma_f})^2}{H_n^2} E\tau_n^2 \\ &\quad + \frac{2C_{f,0}^2}{H_n^2} E \sum_{i=1}^{\tau_n} \left\{ E \left[K \left(\frac{x^0 - x_i}{h_n} \right) | \mathcal{F}_{i-1} \right] - K \left(\frac{x^0 - x_i}{h_n} \right) \right\}^2 \\ &\leq \frac{2(C_1 h_n^{q+1+\gamma_x} + C_2 h_n^{p+\gamma_f})^2 h_n^2 t_n^2}{H_n^2} + \frac{2C_3 C_{f,0}^2 h_n}{H_n^2} E\tau_n \\ &\leq \frac{2(C_1 h_n^{q+1+\gamma_x} + C_2 h_n^{p+\gamma_f})^2 h_n^2 t_n^2}{H_n^2} + \frac{2C_3 C_{f,0}^2}{H_n} (c_x^0 - \kappa h_n^{q+\gamma_x})^{-1} \left(1 + \frac{\bar{K}}{H_n} \right). \end{aligned} \quad (7.10)$$

Then the third assertion of the theorem follows from the definition of the function V_n , (7.9), (7.10) and the inequality

$$E(f_n^* - f)^2 \leq 2[E(f_n^* - \bar{f}_n^*)^2 + E(\bar{f}_n^* - f)^2] \leq V_n.$$

Thus Theorem 3.1 is proved.

7.2. Proof of Theorem 4.1

Consider the deviation of the estimator (4.1):

$$\begin{aligned} f_N^*(h, H) - f &= [\hat{f}_N(h, H) - f] \cdot \chi \left(\sum_{i=1}^N K \left(\frac{x^0 - x_i}{h} \right) \geq H \right) \\ &\quad + \left[\frac{C_{f,0}}{2} - f \right] \cdot \chi \left(\sum_{i=1}^N K \left(\frac{x^0 - x_i}{h} \right) < H \right). \end{aligned} \quad (7.11)$$

Then

$$\begin{aligned} E(f_N^*(h, H) - f)^2 &= E \left[\hat{f}_N(h, H) - f \right]^2 \cdot \chi \left(\sum_{i=1}^N K \left(\frac{x^0 - x_i}{h} \right) \geq H \right) \\ &\quad + \left[\frac{C_{f,0}}{2} - f \right]^2 \cdot P \left(\sum_{i=1}^N K \left(\frac{x^0 - x_i}{h} \right) < H \right) \leq \frac{1}{H^2} E \left[\sum_{i=1}^{\tau_N(h, H)} (Y_i K \left(\frac{x^0 - x_i}{h_n} \right) - f) \right]^2 \\ &\quad + \frac{C_{f,0}^2}{4} \cdot P \left(\sum_{i=1}^N K \left(\frac{x^0 - x_i}{h} \right) < H \right). \end{aligned}$$

Similar to the proof of Theorem 3.1, using the inequalities $E\tau_N(h, H) \leq N$ and $E\tau^2(N, H) \leq N^2$ in (7.8), (7.9) and (7.10) instead of (3.2) and (3.3) for $E\tau_n$ and $E\tau_n^2$ can be proved the inequality:

$$\begin{aligned} & \frac{1}{H^2} E \left[\sum_{i=1}^{\tau_N(h, H)} \left(Y_i K \left(\frac{x^0 - x_i}{h} \right) - f \right) \right]^2 \\ & \leq 2 \left\{ (2C_3 C_{f,0}^2 + C_4(1 + \gamma_{x,\delta})) \frac{hN}{H^2} + (1 + \gamma_{x,\delta}) \left[\varphi(0) \bar{K}^2 \gamma_\delta \frac{l(H)}{H^2} (1 + 3l(H)) \right. \right. \\ & \left. \left. + C_3 \varphi(0) \frac{hN}{H^2} + 2\gamma_\delta \varphi(0) K_0 \frac{l(H)hN}{H^2} + 2\gamma_\delta K_0 \bar{K} \varphi(l(H)) \frac{hN^2}{H^2} \right] + \frac{2(C_1 h^{q+1+\gamma_x} + C_2 h^{p+\gamma_f})^2 h^2 N^2}{H^2} \right\}. \end{aligned}$$

Now, using the Chebyshev inequality, for $Nh(c_x^0 - \kappa h^{q+\gamma_x}) - H > 0$ we estimate the probability

$$\begin{aligned} & P \left(\sum_{i=1}^N K \left(\frac{x^0 - x_i}{h} \right) < H \right) \\ & = P \left(\sum_{i=1}^N \left(EK \left[\left(\frac{x^0 - x_i}{h} \right) | \mathcal{F}_{i-1} \right] - K \left(\frac{x^0 - x_i}{h} \right) \right) > \sum_{i=1}^N EK \left[\left(\frac{x^0 - x_i}{h} \right) | \mathcal{F}_{i-1} \right] - H \right) \\ & \leq P \left(\sum_{i=1}^N \left(EK \left[\left(\frac{x^0 - x_i}{h} \right) | \mathcal{F}_{i-1} \right] - K \left(\frac{x^0 - x_i}{h} \right) \right) > Nh(c_x^0 - \kappa h^{q+\gamma_x}) - H \right) \\ & \leq \frac{1}{(Nh(c_x^0 - \kappa h^{q+\gamma_x}) - H)^2} E \left(\sum_{i=1}^N \left(EK \left[\left(\frac{x^0 - x_i}{h} \right) | \mathcal{F}_{i-1} \right] - K \left(\frac{x^0 - x_i}{h} \right) \right) \right)^2 \\ & \leq \frac{C_3 Nh}{(Nh(c_x^0 - \kappa h^{q+\gamma_x}) - H)^2} \end{aligned}$$

and two first assertions of Theorem 4.1 are proved.

For the proof of the third assertion it is enough to prove the following limit relation:

$$\lim_{N \rightarrow \infty} \frac{1}{H_N} \sum_{i=1}^N K \left(\frac{x^0 - x_i}{h_N} \right) > 1 \quad \text{a.s.} \quad (7.12)$$

To this end we note that, similarly to the beginning of the proof of Theorem 3.1 we have the inequalities

$$\frac{1}{Nh_N} \sum_{i=1}^N EK \left(\frac{x^0 - x_i}{h_N} \right) | \mathcal{F}_{i-1} \geq c_x^0 - \kappa h_{n_0}^{q+\gamma_x} > 0, \quad N \geq n_0. \quad (7.13)$$

Then we prove the following relations

$$\begin{aligned} & E \left| \frac{1}{Nh_N} \sum_{i=1}^N K_{i,N} \right|^r \leq \frac{(8\bar{K}^2 r)^{r/2}}{N^{r/2} h_N^{r-1}}, \quad N \geq 1, \quad (7.14) \\ & K_{i,N} = K \left(\frac{x^0 - x_i}{h_N} \right) - K \left[\left(\frac{x^0 - x_i}{h_N} \right) | \mathcal{F}_{i-1} \right], \end{aligned}$$

using the covariance (Burkholder) inequality

$$E \left| \sum_{i=1}^N X_i \right|^m \leq (2m)^{m/2} \left(\left(E(|X_i| \sum_{j=i}^N |E(X_j|\mathcal{F}_i)|)^{m/2} \right)^{2/m} \right)^{m/2},$$

which holds true, according to Dedecker and Doukhan (1990), for $m \geq 2$ and every sequence $(X_i, \mathcal{F}_i)_{1 \leq i \leq N}$, such that $\max_{1 \leq i \leq N} E|X_i|^m < \infty$.

Indeed, these inequalities are fulfilled for $m = r$ and $X_i = K_{i,N}$ in our case in view of the following relations:

$$E(K_{j,N}|\mathcal{F}_i) = K_{j,N} \cdot \chi(j = i), \quad E\left[\left|K\left(\frac{x^0 - x_i}{h_N}\right)\right|^s |\mathcal{F}_{i-1}]\right] \leq \overline{K}^s h_N, \quad s > 0,$$

C_r -inequality: $|a + b| \leq C_r(|a|^r + |b|^r)$, $C_r = 1$ if $0 < r < 1$; $C_r = 2^{r-1}$ if $r \geq 1$ and

$$\begin{aligned} E|K_{j,N}|^r &\leq 2^{r-1} E\left(\left|K\left(\frac{x^0 - x_i}{h_N}\right)\right|^r + \left|K\left(\frac{x^0 - x_i}{h_N}\right)\right|^r |\mathcal{F}_{i-1}]\right)^r \\ &\leq 2^{r-1} E\left(E\left[\left|K\left(\frac{x^0 - x_i}{h_N}\right)\right|^r |\mathcal{F}_{i-1}]\right] + \left|K\left(\frac{x^0 - x_i}{h_N}\right)\right|^r |\mathcal{F}_{i-1}]\right)^r \leq (2\overline{K})^r h_N. \end{aligned}$$

Thus, from the Borel-Cantelli lemma, condition from the third assertion of Theorem 4.1 and (7.14) we have

$$\lim_{N \rightarrow \infty} \frac{1}{Nh_N} \sum_{i=1}^N K_{i,N} = 0 \quad \text{a.s.}$$

and, using (7.13), we obtain

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{H_N} \sum_{i=1}^N K\left(\frac{x^0 - x_i}{h_N}\right) &= \lim_{N \rightarrow \infty} \frac{C_\alpha}{Nh_N} \sum_{i=1}^N E\left[K\left(\frac{x^0 - x_i}{h_N}\right) |\mathcal{F}_{i-1}]\right] \\ &\geq \frac{c_x^0 - \kappa h_{n_0}^{q+\gamma_x}}{c_x^0 - \kappa h_{n_0}^{q+\gamma_x} - \alpha^{-1/2}} > 1 \quad \text{a.s.} \end{aligned}$$

Theorem 4.1 is proved.

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