
Resampling and Subsampling for Financial Time Series

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1 Introduction

Consider a discrete time process describing the behavior of log-returns

$$R_t = \log \left(1 + \frac{P_t - P_{t-1}}{P_{t-1}} \right), \quad t = 1, 2, \dots$$

where $\{P_t, t = 0, 1, 2, \dots\}$ is the price of a financial asset observed at time t , t can be measured in seconds, minutes, hours, days, etc. Standard examples for P_t are prices of company-shares quoted at major stock exchanges, interest rates and foreign exchange rates among different currencies. A Taylor series argument shows that R_t is close to the relative returns $(P_t - P_{t-1})/P_{t-1}$ which are free of unit and describe the relative change over time of the price process. Statistical inference for such financial time series has received considerable interest in the last decades leading to a vast and growing literature.

A general class of models proposed to describe the behavior of log returns is given by the following multiplicative model

$$R_t = \mu_t + \sigma_t \cdot \varepsilon_t, \tag{1}$$

where μ_t and σ_t are random variables measurable with respect to the σ -field $\mathcal{F}_{t-1} = \sigma(R_{t-j}, j = 1, 2, \dots)$ and $\{\varepsilon_t\}$ denotes a sequence of i.i.d. random variables independent of $\{R_{t-j}, j \geq 1\}$ with zero mean and unit variance. It is also assumed that ε_t is independent of the conditional (on the past log-returns $R_{t-j}, j = 1, 2, \dots$) mean μ_t and of the conditional (non-negative) volatility function σ_t^2 . By model (1) the direction of the price change at time t is determined by the sign of ε_t , while the order of magnitude of this change by the volatility process σ_t^2 which is independent of ε_t . In what follows we assume for simplicity that $\mu_t \equiv 0$ and concentrate on bootstrap-based statistical inference for the conditional variance σ_t^2 of the log-returns.

Several of the statistical models proposed in the literature specify the volatility function σ_t^2 as a function of the observable past values R_{t-j} and

$\sigma_{t-j}^2, j = 1, 2, \dots$. For instance, $\{R_t\}$ follows a general nonnegative ARCH(∞) equation if for some known nonnegative function $w(\cdot)$,

$$w(R_t) = \rho_t \xi_t, \quad (2)$$

where

$$\rho_t = a + \sum_{j=1}^{\infty} \beta_j w(R_{t-1}), \quad (3)$$

$\{\xi_t\}$ is a sequence of i.i.d. nonnegative random variables, $a \geq 0$ and $\beta_j \geq 0$, $j = 1, 2, \dots$; see Robinson (1991), Giraitis et al. (2000) and Kazakevičius and Leipus (2002). $E\xi_1^2 < \infty$ and $\sum_{j=1}^{\infty} \beta_j^2 < 1$ imply weak stationarity of ρ_t . The class (2)-(3) is rich enough and includes as special cases the classical ARCH(p) process (Engle (1982)) as well as the ARCH(∞) process obtained for $w(x) = x^2$,

$$R_t^2 = \left(a + \sum_{j=1}^{\infty} \beta_j R_{t-j}^2 \right) \varepsilon_t^2.$$

Under certain assumptions on the behavior of the b_j 's this class includes also the GARCH(p, q) models, Bollerslev (1986); see also Taylor (1986).

An alternative approach to the above ARCH(∞) model class is the class of stochastic volatility models a simple form of which is given if σ_t in (1) satisfies

$$\sigma_t = g(h_t), \quad \text{with } h_t = a_0 + a_1 h_{t-1} + e_t, \quad (4)$$

where $g(\cdot) > 0$ is a known function and $\{e_t\}$ is an i.i.d. process independent from $\{\varepsilon_t\}$. For $|a_1| < 1$ the process h_t is strictly stationary. Notice that the heteroscedastic variation of R_t described by σ_t^2 is driven by the unobservable latent process $\{h_t\}$ and not by the lagged values of the log-returns R_{t-1}, R_{t-2}, \dots . Thus for this model class, the volatility function is modeled as a strictly stationary process $\{\sigma_t^2\}$ independent of the i.i.d. noise process $\{\varepsilon_t\}$ avoiding therefore any kind of feedback between the noise and the volatility process; cf. Shepard (1996).

Given a time series R_1, R_2, \dots, R_n of log-returns, one is commonly interested in the construction of point or interval estimators of the volatility function σ_t^2 and in testing hypothesis about this function. Several estimation methods have been proposed in the literature depending on the assumptions imposed on σ_t^2 . They rank from parametric methods designed when σ_t^2 belongs to certain parametric classes of models to fully nonparametric methods based on weak assumptions on the function of interest. In this context, nonparametric procedures are useful not only because they lead to estimators of the underlying volatility function without imposing too restrictive assumptions but also because they are very useful for model selection and testing by means of comparing parametric and nonparametric estimates; cf. for instance Kreiss et al. (2004).

Assigning properties of estimators of σ_t^2 is usually carried out by means of asymptotic considerations where the expressions obtained for the asymptotic quantities usually depend in a complicated way on characteristics of the underlying process. This makes alternative approaches based on bootstrap

methodology appealing. During the last decades different bootstrap methods have been proposed in the context of financial time series. Some early application of the bootstrap in financial time series mainly based on i.i.d. resampling (cf. Maddala and Li (1996) and Ruiz and Pascual (2002) for a review) are not appropriate and may lead to wrong conclusions since log-returns are not independent despite their vanishing correlation. Thus approaches to bootstrap financial time series should take into account their dependence structure or at least those aspects of this dependence structure which are important for the particular inference problem at hand.

Nonparametric methods to bootstrap time series which are based on resampling with replacement from blocks of consecutive observations can in principle be applied to financial series, and their properties to approximate the distribution of statistics of interest can be investigated provided the underlying stochastic process obeys some appropriate weak dependence, e.g., mixing conditions. For mixing properties of some commonly used time series models with applications in finance, see for instance, Carrasco and Chen (2002). Under appropriate mixing conditions, bootstrapping a series of log-returns R_1, R_2, \dots, R_n can be done by randomly choosing with replacement a number of l , $l = \lfloor n/b \rfloor$, blocks of b consecutive values $\{R_t, R_{t+1}, \dots, R_{t+b-1}\}$ from all possible $n - b$ blocks; cf. Künsch (1989), Liu and Sign (1992), Politis and Romano (1994); see also Bühlmann (2002) and Härdle et al. (2003) for an overview and the monograph by Lahiri (2003). Although such general blocking techniques preserve the dependence structure of the observations within blocks, they have not been widely used in the context of financial time series. A reason for this might be that since it is common to impose some kind of model structure in describing the behavior of financial time series, efficiency considerations make model-based bootstrap methods more attractive. Furthermore, and concerning nonparametric estimators of the volatility function σ_t^2 , it is well-known that the dependence structure of the underlying process affects the limiting behavior of the estimators only through the behavior of certain finite dimensional, stationary distributions of the process. This again suggests that bootstrapping nonparametric estimators for financial time series can be successfully done without mimicking the whole and probably very complicated dependence structure of the observed series.

In Section 2 we review different methods to bootstrap financial time series, that is methods that can be applied to generate pseudo-replicates $R_1^*, R_2^*, \dots, R_n^*$ of the observed series of log-returns. Such methods are designed in a way that mimics closely the essential dependence characteristics of the observed time series or at least those characteristics which are important for inferring consistently properties of the statistics of interest. In Section 3 we concentrate on the somewhat different problem on how to apply the bootstrap in order to infer properties of statistics based on financial time series. Such methods do not work by generating new pseudo-observations of the observed log-returns that preserves their dependence structure, but by generating pseudo-replicates of the statistic of interest. Section 4 is devoted to subsampling and self-normalization methods applied to financial data.

2 Resampling the Time Series of Log>Returns

The bootstrap procedures described in this section generate replications $R_1^*, R_2^*, \dots, R_n^*$ of the series of log-returns. Bootstrap-based inference is then provided by approximating properties of the statistics based on the original time series R_1, R_2, \dots, R_n by the corresponding properties of the statistics based on the bootstrap replicates $R_1^*, R_2^*, \dots, R_n^*$.

2.1 Parametric methods based on i.i.d. resampling of residuals

The basic idea to bootstrap a series of log-returns when parametric assumptions on the volatility function $\sigma_t^2(\cdot)$ are imposed, is to obtain residuals using the estimated parametric model and to generate new pseudo-series of log-returns using the estimated model structure and i.i.d. resampling of residuals. More specifically, suppose that

$$R_t = \sigma_t(\theta)\varepsilon_t \quad (5)$$

where the function $\sigma_t^2(\theta)$ belongs to some parametric family of functions with $\theta \in \Theta$ and Θ a finite dimensional parameter space. Furthermore, $\{\varepsilon_t\}$ denotes a sequence of i.i.d. random variables with zero mean and unit variance.

As an example consider a GARCH(p,q) specification of $\sigma_t^2(\theta)$ given by

$$\sigma_t^2(\theta) = c + \sum_{i=1}^p a_i R_{t-i}^2 + \sum_{j=1}^q b_j \sigma_{t-j}^2, \quad (6)$$

$\theta = (c, a_1, \dots, a_p, b_1, \dots, b_q)'$, $c > 0$, $a_i \geq 0$, $b_j \geq 0$ and p and q nonnegative integers. Strictly stationarity of the stochastic process $\{X_t, t \in \mathbb{Z}\}$ with $E(X_t^2) < \infty$ follows if $\sum_{i=1}^p a_i + \sum_{j=1}^q b_j < 1$, Bollerslev (1986); see also Bougerol and Picard (1992). Let $\{\hat{\theta} = \hat{\theta}_n, n \in \mathbb{N}\}$ be a sequence of estimators $\hat{\theta} = (\hat{c}, \hat{a}_1, \dots, \hat{a}_p, \hat{b}_1, \dots, \hat{b}_q)'$ of θ , for instance, the commonly used conditional maximum likelihood estimator. Define standardized residuals

$$\hat{\varepsilon}_t = \frac{\tilde{\varepsilon}_t - \tilde{n}^{-1} \sum_i \tilde{\varepsilon}_i}{\{\tilde{n}^{-1} \sum_i \tilde{\varepsilon}_i^2 - (\tilde{n}^{-1} \sum_i \tilde{\varepsilon}_i)^2\}^{1/2}},$$

where $\tilde{\varepsilon}_t = R_t/\sigma_t(\hat{\theta})$ are the estimated model residuals and $\tilde{n} = n - p$. Pseudo-series of log returns following a GARCH(p,q) model structure can then be generated using the equation

$$R_t^* = \sigma_t(\hat{\theta})\varepsilon_t^*,$$

where $\sigma_t(\hat{\theta})$ is the specification (6) of the conditional variance function with θ replaced by its estimator $\hat{\theta}$, $\{\varepsilon_t^*\}$ is an i.i.d. sequence with $\varepsilon_t^* \sim F_n$ and F_n is the empirical distribution function of the $\hat{\varepsilon}_t$'s.

In most applications of such a parametric bootstrap procedure it is assumed that ε_t has finite fourth moments, i.e., $E\varepsilon_t^4 < \infty$. For instance and for ε_t being standard Gaussian errors, Kokoszka et al. (2004) use this bootstrap method to construct confidence intervals for the autocorrelations of the

squares of log-returns, while Miguel and Olave (1999) considered parametric bootstrap prediction intervals for ARCH processes. The assumption of finite fourth moments is crucial for proving consistency of such a parametric bootstrap, since, loosely speaking, this assumption ensures asymptotic normality of the estimator involved with an appropriate, \sqrt{n} -rate of convergence. Without such a moment assumption consistency of this parametric bootstrap procedure might be questionable.

For instance, let $\hat{\theta}^*$ be the estimator of θ based on the pseudo-returns $R_1^*, R_2^*, \dots, R_n^*$ and consider the problem of estimating the distribution of an appropriately rescaled version of $\hat{\theta} - \theta$ by the corresponding distribution of the bootstrap estimator $\hat{\theta}^* - \hat{\theta}$. In such a context, consistency of the parametric bootstrap procedure depends on the limiting behavior of $\hat{\theta} - \theta$ which in turn depends on the distribution of the i.i.d. errors ε_t . If $E(\varepsilon_1)^4 < \infty$ then it has been shown by Berkes et al. (2003) that under some regularity conditions, $\sqrt{n}(\hat{\theta} - \theta) \Rightarrow N(0, (E\varepsilon_1^4 - 1)\Sigma_\theta^{-1})$ as $n \rightarrow \infty$, where $\Sigma_\theta = E(\sigma_1^{-4}(\theta)U(\theta)U'(\theta))$ and $U(\theta)$ is the r -dimensional vector of first derivatives of $\sigma_1^2(\theta) = \sigma_1^2(a, b, c)$ with respect to the components of $a = (a_1, \dots, a_p), b = (b_1, \dots, b_q)$ and c , evaluated at θ . The asymptotic normality of $\sqrt{n}(\hat{\theta} - \theta)$ in this case, suggests the use of the distribution of $\sqrt{n}(\hat{\theta}^* - \hat{\theta})$ to approximate the distribution of the former statistic. However, if $E(\varepsilon_1^4) = \infty$ then the normalizing sequence of the estimation error $\hat{\theta} - \theta$ is no more \sqrt{n} and the limiting distribution of the appropriately normalized sequence $\hat{\theta} - \theta$ differs from a Gaussian law and depends on the distribution of the squared errors ε_1^2 ; see Straumann (2005) and Hall and Yao (2003). In such situations, subsampling offers an alternative; cf. Politis et al. (1999). A subsampling based approach to infer properties of parameter estimators for GARCH processes which is asymptotically valid under general assumptions on the distribution of the errors ε_t has been proposed by Hall and Yao (2003); see Section 4 for details.

2.2 Nonparametric methods based on i.i.d. resampling of residuals

If instead of a parametric structure for $\sigma(\cdot)$ the general nonparametric model

$$R_t = \sigma(R_{t-1}, R_{t-2}, \dots, R_{t-p})\varepsilon_t, \quad (7)$$

is assumed, then the volatility function $\sigma^2(\cdot)$ can be estimated in a nonparametric way and pseudo-series of log-returns can be generated using the estimated volatility function and i.i.d. resampling of estimated residuals. Meaningful statistical inference for $\sigma^2(\cdot)$ in a nonparametric context, requires however, that the discrete time Markov process $\{R_t\}$ following (7) satisfies some stability and ergodicity conditions which can be achieved by imposing some restrictions on the distribution of the errors ε_t and on the function $\sigma : R^p \rightarrow (0, \infty)$. For instance, when $p = 1$, $\{R_t\}$ is geometrically ergodic if the density f_ε of ε is absolutely continuous and positive everywhere, $\inf_{x \in R} \sigma(x) > 0$, σ is bounded over bounded sets and satisfies $\limsup_{|x| \rightarrow \infty} E|\sigma(x)\varepsilon_1|/|x| < 1$; cf. Franke et al. (2002b).

Let $\hat{\sigma}^2(x_1, \dots, x_p)$ be a nonparametric estimator of $\sigma^2(x_1, \dots, x_p) = E(R_t^2 | R_{t-1} = x_1, \dots, R_{t-p} = x_p)$ and define standardized residuals

$$\widehat{\varepsilon}_t = \frac{\widetilde{\varepsilon}_t - \widetilde{n}^{-1} \sum_i \widetilde{\varepsilon}_i}{\{\widetilde{n}^{-1} \sum_i \widetilde{\varepsilon}_i^2 - (\widetilde{n}^{-1} \sum_i \widetilde{\varepsilon}_i)^2\}^{1/2}},$$

where $\widetilde{\varepsilon}_t = R_t/\widehat{\sigma}(R_{t-1}, R_{t-2}, \dots, R_{t-p})$, $t = p+1, p+2, \dots, n$ and $\widetilde{n} = n-p$. Pseudo-series of log returns can then be generated as

$$R_t^* = \widehat{\sigma}(R_{t-1}^*, \dots, R_{t-p}^*)\varepsilon_t^*, \quad (8)$$

where the ε_t^* 's are i.i.d. random variables with $\varepsilon_t^* \sim F_n$ and F_n is the empirical distribution function of the $\widehat{\varepsilon}_t$; cf. Franke et al. (2002a).

To fix ideas, let $p = 1$ and consider the following nonparametric estimator of the volatility function

$$\widehat{\sigma}_h^2(x) = \frac{1}{\widehat{f}_{R_t}(x)} \frac{1}{(n-1)} \sum_{t=1}^{n-1} K_h(x - R_t)(R_{t+1} - \overline{R}_n)^2, \quad (9)$$

where h is a smoothing bandwidth, K a smoothing kernel and $\overline{R}_n = n^{-1} \sum_{t=1}^n R_t$. Notice that centering by the sample mean \overline{R}_n is used since it is assumed that $\mu_t = 0$. If the conditional mean function μ_t is not constant and $\mu_t = m(R_{t-1})$ with $m(\cdot)$ some smooth function, then \overline{R}_n should be replaced by a nonparametric estimator $\widehat{m}_h(R_{t-1})$ of the conditional mean function $m(R_{t-1}) = E(R_t|R_{t-1})$; see Fan and Yao (1998) for details and for a comparison of different estimators of the volatility function.

The bootstrap procedure described above can be applied to approximate the distribution of random variables like $\sqrt{nh}(\widehat{\sigma}_h^2(x) - \sigma^2(x))$ and $\sup_{x \in [a,b]} |\widehat{\sigma}_h^2(x) - \sigma^2(x)|$, for some $a < b$, or of some standardized versions thereof. Franke et al (2002b) established under certain regularity conditions absolute regularity and geometric ergodicity of the bootstrap process $\{R_t^*, t \in Z\}$ and applied these results to show validity of the corresponding nonparametric bootstrap procedure in approximating the distribution of supremum type statistics of the conditional mean estimator. Franke et al. (2000a) consider applications to pointwise statistics while Franke et al. (2004) investigated properties of such a bootstrap procedure for the construction of confidence bands for the volatility function.

Notice that an inherent problem common in applications of the bootstrap to estimate the distribution of nonparametric estimators is how to deal with the bias. In particular, and decomposing the statistic of interest in a stochastic and in a bias term, i.e., writing

$$\sqrt{nh}(\widehat{\sigma}_h^2(x) - \sigma^2(x)) = \sqrt{nh}(\widehat{\sigma}_h^2(x) - E(\widehat{\sigma}_h^2(x))) + \sqrt{nh}(E(\widehat{\sigma}_h^2(x)) - \sigma^2(x)),$$

makes it clear that if one is interested in applying the bootstrap in order to approximate the distribution of $\sqrt{nh}(\widehat{\sigma}_h^2(x) - \sigma^2(x))$, then such a bootstrap procedure should approximate correctly the behavior of both terms on the right hand side of the above decomposition. In this context correct estimation of the bias term requires a kind of oversmoothing, i.e., the use of an estimator $\widehat{\sigma}_g^2(x)$ to generate the pseudo-variables R_t^* 's in (8) for which the bandwidth g satisfies $g > h$ and $h/g \rightarrow 0$ as $n \rightarrow \infty$. The bootstrap statistic used to approximate the distribution of $\sqrt{nh}(\widehat{\sigma}_h^2(x) - \sigma^2(x))$ is then given by

$\sqrt{nh}(\hat{\sigma}_h^*(x) - \sigma_g^2(x))$ where $\hat{\sigma}_h^*(x)$ is the estimator (9) with R_t replaced by the bootstrap pseudo-variables R_t^* and \bar{R}_n by $\bar{R}_n^* = n^{-1} \sum_{t=1}^n R_t^*$. An alternative to such a bootstrap estimation of the bias would be explicit bias correction; see Hall(1992) for a discussion of this issue in the context of independent data.

Although the above nonparametric bootstrap procedure can be successfully applied to estimate the distribution of the nonparametric estimators of interest, it fails if the process of log-returns does not follow model (7). The reason for this is that in this case the stationary distribution of the bootstrap variables $(R_t^*, R_{t-1}^*, \dots, R_{t-p}^*)$ does not converge to the desired stationary distribution of $(R_t, R_{t-1}, \dots, R_{t-p})$. In fact and for $\mathbf{R}_{t-1, t-p} = (R_{t-1}, R_{t-2}, \dots, R_{t-p})$, we expect is such a case that the conditional distribution function $P(R_t^* \leq \cdot | \mathbf{R}_{t-1, t-p}^* = \mathbf{x})$ of the bootstrap process will behave asymptotically like $F_{U_t}(\cdot / \sigma(\mathbf{x}))$ where F_{U_t} denotes the stationary distribution function of $U_t = R_t / \sigma(\mathbf{x})$ and $\sigma(\mathbf{x}) = \sqrt{\sigma^2(\mathbf{x})}$ with $\sigma^2(\mathbf{x}) = E(R_t^2 | R_{t-i} = x_i, i = 1, 2, \dots, p)$. This distribution differs however from the conditional distribution function $P(R_t \leq \cdot | \mathbf{R}_{t-1, t-p} = \mathbf{x})$ of the underlying process if model (7) is not correct.

2.3 Markovian bootstrap

Model (7) is a special case of a more general p -th order Markovian process, that is a process $\{R_t, t \in Z\}$ which satisfies

$$P(R_t \in A | \sigma(R_s, s < t)) = P(R_t \in A | R_{t-1}, R_{t-2}, \dots, R_{t-p}),$$

for all $A \in \mathcal{B}(R)$ and all $t \in Z$. Suppose that such a model describes the behavior of log-returns and denote by $F_{R_t | \mathbf{R}_{t-1, t-p}}(\cdot | \mathbf{x}) = P(R_t \leq y | R_{t-i} = x_i, i = 1, 2, \dots, p)$ the one-step transition distribution function. Assume that the corresponding conditional probability measure possesses a density with respect to Lebesgue measure which we denote by $f_{R_t | \mathbf{R}_{t-1, t-p}}(\cdot | \mathbf{x})$. Notice that model (7) is a specific Markov process with one-step transition distribution function given by $F_{R_t | \mathbf{R}_{t-1, t-p}}(y | \mathbf{x}) = F_\varepsilon(y / \sigma(\mathbf{x}))$ where F_ε is the distribution function of the error ε_1 .

Imposing conditions on $F_{R_t | \mathbf{R}_{t-1, t-p}}(\cdot)$ which ensure stationarity and geometric ergodicity of the associated Markov chain, bootstrap replicates of log-returns can be generated using the Markovian model structure without specifying its functional form. Rajarshi (1990) proposed such a bootstrap approach based on a nonparametric estimator $\hat{f}_{R_t | \mathbf{R}_{t-1, t-p}}(\cdot | \mathbf{x})$ of the one step transition density $f_{R_t | \mathbf{R}_{t-1, t-p}}(\cdot | \mathbf{x})$. Such a nonparametric estimator is for instance given by

$$\hat{f}_{R_t | \mathbf{R}_{t-1, t-p}}(y | \mathbf{x}) = \sum_{t=p+1}^n K_b((y, \mathbf{x}) - \mathbf{R}_{t, t-p}) / \sum_{t=p+1}^n K_b(\mathbf{x} - \mathbf{R}_{t-1, t-p}),$$

where b is the bandwidth used to estimate the stationary densities of interest. New series of pseudo-replications can then be generated as

$$R_t^* \sim \hat{f}_{R_t | \mathbf{R}_{t-1, t-p}}(\cdot | R_{t-1}^*, R_{t-2}^*, \dots, R_{t-p}^*).$$

A different approach which does not require explicit nonparametric estimation and resamples directly the original series of log-returns in an appropriate way preserving their Markovian dependence structure, has been proposed by Paparoditis and Politis (2001a). Given a series $R_{t-p}^*, R_{t-p+1}^*, \dots, R_{t-1}^*$ of pseudo log-returns, their approach works by generating a new pseudo-variable R_t^* as

$$R_t^* = R_J,$$

where J is a discrete random variable taking values in the set $\{p+1, p+2, \dots, n\}$ with

$$P(J = s) = \frac{W_b(\mathbf{R}_{t-1, t-p}^* - \mathbf{R}_{s-1, s-p})}{\sum_{l=p+1}^n W_b(\mathbf{R}_{t-1, t-p}^* - \mathbf{R}_{l-1, l-p})}$$

for $s \in \{p+1, p+2, \dots, n\}$, where $\mathbf{R}_{t-1, t-p}^* = (R_{t-p}^*, R_{t-p+1}^*, \dots, R_{t-1}^*)$. Here $W_b(\cdot) = b^{-p}W(\cdot/p)$, where b is the so-called resampling width and $W(\cdot)$ a p -dimensional, nonnegative and symmetric resampling kernel with mean zero. Notice that this procedure resamples the observed log-returns in a way according to which the probability of R_s being selected is higher the closer is its preceding segment $(R_{s-1}, R_{s-2}, \dots, R_{s-p})$ to the last generated bootstrap segment $(R_{t-1}^*, R_{t-2}^*, \dots, R_{t-p}^*)$.

Properties of Markovian bootstrap procedures have been investigated by Rajarshi (1990) and Paparoditis and Politis (2001a); see also Horowitz (2003). Applications of such a bootstrap procedure in order to approximate the distribution of nonparametric conditional moment estimators and for constructing pointwise confidence intervals have been investigated by Paparoditis and Politis (2002). Notice that in approximating correctly the bias of the nonparametric estimators involved, a kind of oversmoothing condition is needed here as well leading to some restrictions on the behavior of the resampling width b compared to the smoothing bandwidth h . In particular, b should satisfy $b > h$ and $b/h \rightarrow 0$ as $n \rightarrow \infty$.

We stress here the fact that for the kind of nonparametric estimators discussed in this paper, the range of applicability of the above Markovian procedures goes far beyond the Markov process class. This is due to the fact that the Markovian resampling schemes described above mimics correctly the $(p+1)$ -dimensional stationary distribution of $(R_t, R_{t-1}, \dots, R_{t-p})$ even if the underlying process is not Markov. This property suffices to establish consistency of the above Markovian bootstrap procedures applying to estimate the distribution of the nonparametric estimators of interest for a very broad class of stochastic processes; see Paparoditis and Politis (2002) for details.

3 Resampling Statistics Based on the Time Series of Log-Returns

In the context of dependent data it is possible in certain situations to apply the bootstrap to some statistics of interest without generating new time series of pseudo-observations that preserve the dependence structure of the observed time series. Such applications resample directly the statistic of interest in a

way which mimics correctly those characteristics of the dependence structure of the underlying process which are essential for the random behavior of the statistic of interest.

Suppose for instance, that we are interested in approximating the pointwise distribution of $\sqrt{nh}(\hat{\sigma}_h^2(x) - \sigma^2(x))$ by means of the bootstrap. As it has been already stressed, for a bootstrap procedure to be successful in approximating correctly the (limiting) distribution of this statistic, it is not necessary to imitate the whole and probably very complicated dependence structure of the underlying process of log-returns. For this, it suffices to mimic correctly the $(p+1)$ -dimensional stationary distribution of $(R_t, R_{t-1}, \dots, R_{t-p})$. This is a consequence of the basic fact that the asymptotic distribution of nonparametric estimators does not reflect the dependence structure of the underlying process beyond the $(p+1)$ -dimensional structure; cf. Robinson (1983). Hart (1995) called this the whitening by windowing effect. This basic observation has motivated the development of bootstrap procedures that generate pseudo-replicates $(R_t^*, R_{t-1}^*, \dots, R_{t-p}^*)$ of $(R_t, R_{t-1}, \dots, R_{t-p})$ in a way that mimic correctly the joint distribution of the last random vector, without generating new pseudo-series of log-returns.

3.1 Regression Bootstrap

The regression bootstrap is a nonparametric bootstrap procedure which generates replicates of the pairs $\{(R_t, \mathbf{R}_{t-1,t-p}), t = p+1, p+2, \dots, n\}$ denoted by $\{(R_t^*, \mathbf{R}_{t-1,t-p}^*), t = p+1, p+2, \dots, n\}$ by using a fixed design, heteroscedastic regression model with errors obtained by i.i.d. resampling from estimated model residuals; cf. Franke et al. (2002a). Notice that in this resampling scheme only the random variable R_t is bootstrapped while $\mathbf{R}_{t-1,t-p}$ is treated as a (conditionally) fixed design. In particular, the bootstrap variables R_t^* are generated using the equation

$$R_t^* = \hat{\sigma}_g(R_{t-1}, R_{t-2}, \dots, R_{t-p})\varepsilon_t^*, \quad (10)$$

where the ε_t^* are independent random variables such that $\varepsilon_t^* \sim F_n$ and F_n is the empirical distribution function of the estimated errors $\hat{\varepsilon}_t$ given in Section 2.2. Notice that the random variables R_t^* are (conditional on the observed series) independent with $E^*(R_t^*) = 0$ and $Var^*(R_t^*) = \hat{\sigma}_g^2(R_{t-1}, R_{t-2}, \dots, R_{t-p})$. Thus the dependence structure of the series of log returns is not preserved by this bootstrap method. The distribution of $\sqrt{nh}(\hat{\sigma}_h^2(\mathbf{x}) - \sigma^2(\mathbf{x}))$ can now be approximated by that of $\sqrt{nh}(\hat{\sigma}_h^{*2}(\mathbf{x}) - \hat{\sigma}_g^2(\mathbf{x}))$, where, for $p = 1$,

$$\hat{\sigma}_h^{*2}(x) = \frac{1}{\hat{f}_{R_t}(x)} \frac{1}{(n-1)} \sum_{t=1}^{n-1} K_h(x - R_t)(R_{t+1}^* - \bar{R}_n^*)^2,$$

and $\bar{R}_n^* = n^{-1} \sum_{t=1}^n R_t^*$. The bandwidth g used in (10) can be chosen so that the above bootstrap procedure estimates also correctly the bias term $\sqrt{nh}(E(\hat{\sigma}_h^2(x)) - \sigma^2(x))$ of the nonparametric volatility function estimator. For this an oversmoothing type condition should be satisfied, i.e., $g > h$ with $h/g \rightarrow 0$ as $n \rightarrow \infty$; cf. Franke et al. (2002a).

3.2 Wild bootstrap

The wild bootstrap methodology can be also applied in the context of financial time series to estimate the distribution of some nonparametric estimators of interest; see Franke et al. (2002a) and Kreiss (2000). To fix ideas, let $p = 1$ and consider the centered log-returns $Y_t = R_t - \bar{R}_n$, $t = 1, 2, \dots, n$. Let further η_t , $t = 1, 2, \dots, n$ be a sequence of independent, identically distributed random variables satisfying $E(\eta_t) = 0$, $E(\eta_t^2) = 1$. For higher order performance the distribution of η_t is often chosen such that additionally the condition $E(\eta_t^3) = 1$ is satisfied; cf. Mammen (1992) for a discussion.

The wild bootstrap works by generating pairs $\{(Y_{t+1}^*, R_t), t = 1, 2, \dots, n-1\}$ where

$$Y_{t+1}^* = \hat{\sigma}_g^2(R_t) + \varepsilon_{t+1}^*, \quad (11)$$

and

$$\varepsilon_{t+1}^* = \left[Y_{t+1}^2 - \hat{\sigma}_h^2(R_t) \right] \cdot \eta_{t+1} \quad (12)$$

Notice that $E^*(Y_{t+1}^*) = \hat{\sigma}_g^2(R_t)$ and $Var^*(Y_{t+1}^*)^2 = (Y_{t+1}^2 - \hat{\sigma}_h^2(R_t))^2$. Furthermore, the bootstrap random variables Y_{t+1}^* are (conditional on the observed sample) independent, i.e., the dependence structure of the observed time series is not preserved by this bootstrap scheme. In fact, in the bootstrap world the Y_{t+1}^* 's are generated according to a fixed design nonparametric regression model with mean $\hat{\sigma}_g^2(R_t)$ and variance $Var^*(\varepsilon_{t+1}^*)$. Now, to approximate the distribution of $\sqrt{nh}(\hat{\sigma}_h^2(x) - \sigma^2(x))$ the wild-bootstrap statistic $\sqrt{nh}(\hat{\sigma}_h^{*2}(x) - \sigma_g^2(x))$ can be used, where

$$\hat{\sigma}_h^{*2}(x) = \frac{1}{\hat{f}_{R_t}(x)} \frac{1}{(n-1)} \sum_{t=1}^{n-1} K_h(x - R_t)(Y_{t+1}^*)^2.$$

Notice that in order to capture correctly also the bias term of the above nonparametric estimation, and similar to all bootstrap approaches discussed so far, an oversmoothing type condition is needed by the wild bootstrap as well; see Franke et al. (2002a) and Kreiss (2000).

Although the wild bootstrap does not preserve the dependence structure of the observed series, it resamples correctly the distribution of the nonparametric statistics of interest. This bootstrap scheme is robust against model misspecifications, at least as far as the estimation of the distribution of pointwise statistics like $\sqrt{nh}(\hat{\sigma}_h^2(\mathbf{x}) - \sigma^2(\mathbf{x}))$ is concerned; cf. Kreiss (2000). Neumann and Kreiss (1998) applied a wild bootstrap to the construction of uniform confidence bands for the conditional mean based on supremum type statistics and using strong approximation results under the assumption of a Markovian model. Kreiss (2000) considered the problem of estimating by means of a wild bootstrap procedure the distribution of the sup-distance involved in the construction of simultaneous confidence intervals for the volatility function. His approach is also based on the assumption that the underlying process obeys a Markovian dependence structure.

3.3 Local bootstrap

Another simple and totally model free way to bootstrap nonparametric estimators in time series is the so-called local bootstrap; see Shi (1991) for the case of i.i.d. data and Paparoditis and Politis (2000) for the case of dependent data. This bootstrap method generates replicates $\{(R_t^*, \mathbf{R}_{t-1, t-p}); t = p+1, p+2, \dots, n\}$ of the observed pairs $\{(R_t, \mathbf{R}_{t-1, t-p}); t = p+1, p+2, \dots, n\}$ by correctly imitating the conditional distribution $F_{R_t, \mathbf{R}_{t-1, t-p}}(\cdot | \mathbf{x})$ of the log-returns. In contrast to the regression bootstrap, the local bootstrap resamples the observed values R_t by giving more resampling weights to the values R_s for which $\mathbf{R}_{s-1, s-p}$ is close to $\mathbf{R}_{t-1, t-p}$.

For $p = 1$ and for the problem of estimating the distribution of the nonparametric estimator (9) of the conditional variance function, this procedure can be described as follows. Let $p = 1$ and $Y_t = R_t - \bar{R}_n$, $t = 1, 2, \dots, n$. Bootstrap replicates $\{(Y_{t+1}^*, R_t), t = 1, 2, \dots, n-1\}$ of the pairs $\{(Y_{t+1}, R_t), t = 1, 2, \dots, n-1\}$ are then generated so that the bootstrap pseudo-variable Y_{t+1}^* satisfies

$$P(Y_{t+1}^* = Y_{s+1} | R_t) = \frac{W_b(R_t - R_s)}{\sum_{l=1}^{n-1} W_b(R_t - R_l)}.$$

Here b is the so-called resampling width which determines the neighborhood from which replicates of Y_{t+1} are selected and $W(\cdot)$ a resampling kernel. It is easily seen that

$$E^*((Y_{t+1}^*)^2 | R_t = x) = \frac{1}{\widehat{f}_{R_t, b}(x)} \frac{1}{(n-1)} \sum_{t=1}^{n-1} W_b(x - R_t) (R_{t+1} - \bar{R}_n)^2,$$

i.e., $E^*((Y_{t+1}^*)^2 | R_t = x) = \widehat{\sigma}_b^2(x)$. This suggests that the distribution of $\sqrt{nh}(\widehat{\sigma}_h^2(x) - \sigma^2(x))$ can be approximated by that of $\sqrt{nh}(\widehat{\sigma}_h^{*2}(x) - \widehat{\sigma}_b^2(x))$ where

$$\widehat{\sigma}_h^{*2}(x) = \frac{1}{\widehat{f}_{R_t, h}(x)} \frac{1}{(n-1)} \sum_{t=1}^{n-1} K_h(x - R_t) (Y_{t+1}^*)^2.$$

Consistency properties of such a bootstrap procedure for estimating the distribution of pointwise statistics based on nonparametric estimators of conditional moments for time series data have been established by Paparoditis and Politis (2000). Ango Nze et al. (2002) used such a procedure to estimate the distribution of nonparametric moment estimators under weak dependent assumptions (see Doukhan and Louchichi (1999)).

4 Subsampling and Self-Normalization

Subsampling for dependent data is a method valid in extreme generality. To define it, let $\hat{\theta}_n = \hat{\theta}_n(R_1, \dots, R_n)$ be an arbitrary statistic that is consistent for a general parameter θ at rate a_n , i.e., for large n , the law of $a_n(\hat{\theta}_n - \theta)$ tends to some well-defined asymptotic distribution J . The rate a_n does not have to equal \sqrt{n} , and the distribution J does not have to be normal; we do not even need to know its shape, just that it exists. Let $\hat{\theta}_{i,b} = \hat{\theta}_b(R_i, \dots, R_{i+b-1})$ be the

subsample value of the statistic computed from the i th block of length b . The subsampling estimator of J is $\hat{J}_{b,SUB}$ defined as the empirical distribution of the normalized (and centered) subsample values $a_b(\hat{\theta}_{i,b} - \hat{\theta}_n)$ for $i = 1, \dots, q$ where $q = n - b + 1$.

The asymptotic consistency of $\hat{J}_{b,SUB}$ for general statistics under very weak conditions was shown in Politis and Romano (1994); consequently, confidence intervals and/or tests for θ can immediately be formed using the quantiles of $\hat{J}_{b,SUB}$ instead of the quantiles of the (unknown) J . Notice that if only a variance estimator for $a_n\hat{\theta}_n$ is sought, it can be constructed by the sample variance of the normalized subsample values $a_b\hat{\theta}_{i,b}$ for $i = 1, \dots, q$; consistency of the subsampling estimator of variance was shown by Carlstein (1986) under some uniform integrability conditions.

In addition to the usual requirement $b \rightarrow \infty$ as $n \rightarrow \infty$ but with $b/n \rightarrow 0$, the conditions of Politis and Romano (1994) boil down to:

- C1. The series $\{R_t\}$ is strictly stationary.
- C2. The series $\{R_t\}$ is strong mixing.
- C3. The rate a_n is known.

Notice that condition C3 is important for the practical construction of confidence intervals and tests, not for the consistency of subsampling as a method.

Although Conditions C1-C3 are quite weak, they manage to exclude a number of interesting settings pertaining to financial time series since the latter are often plagued by heteroscedasticity, long-range dependence, and heavy tails. Fortunately, all the above conditions can still be relaxed. A review of non-standard conditions for which (variations of) the block bootstrap are consistent can be found in Politis (2003) together with a discussion on the important issue of block size choice; see also the monograph by Lahiri (2003). Condition C1 can easily be relaxed to just asymptotic stationarity as enjoyed, for example, by Markov processes that are generated with an arbitrary (non-equilibrium) start-up distribution; see Politis et al. (1999, Ch. 4). The strong mixing condition C2 has been recently relaxed to the weak dependence condition of Doukhan and Louhichi (1999); see Ango Nze et al. (2003). Finally, condition C3 was relaxed since Bertail et al. (1999) showed how to construct subsampling-based estimators of the rate a_n that can be employed for the construction of confidence intervals and tests.

The above can be seen as small improvements/perturbations on the original Conditions C1-C3. We now describe in more detail how to address major break-downs of those conditions using two broad techniques: (a) examination of the time series of subsample values $\hat{\theta}_{i,b}$ for $i = 1, \dots, q$, and (b) the idea of self-normalization.

To elaborate on using technique (a) in order to relax Condition C2, consider the familiar situation of a unit-root test such as the set-up in Paparoditis and Politis (2003). Under the unit-root hypothesis the data are not strong mixing, not even weakly dependent; however, the subsample values of the Phillips-Perron (say) unit-root statistic *are* weakly dependent and thereby the validity of subsampling is preserved; see Politis et al. (1999, Theorem 12.2.1), Politis et al. (2004), and Romano and Wolf (2001).

The same idea can be applied to relax Condition C1. For example, consider a series that is not strictly stationary, e.g., heteroskedastic. If the subsample values of the pertinent statistic have distributions that converge to the limit J in a uniform way, then subsampling remains consistent; see Politis et al. (1997). An important example is least squares regression with dependent errors and/or covariates; subsampling is shown to work here under conditions *without* assuming a stationary or homoskedastic error structure (Politis et al. 1997, Theorem 3.4).

The idea of self-normalization is closely related to that of studentization. To describe it, we now focus on a particular example that may defy Condition C3: assume Condition C1, and consider the simple case where the statistic of interest $\hat{\theta}_n = n^{-1} \sum_{t=1}^n R_t$ is the sample mean which is an estimator of the expected value $\theta = E(R_t)$ (assumed to be finite). As previously mentioned, the rate a_n is not necessarily \sqrt{n} . Two major avenues resulting in a rate that is less than \sqrt{n} are heavy tailed data, and/or long-range dependence.

To fix ideas, suppose that $a_n = n^\alpha$ for some unknown $\alpha \in (0, 1]$. A successful self-normalization entails constructing a positive statistic, say $\hat{\zeta}_n > 0$, that converges to some limit distribution Q at a rate explicitly related to the unknown α . For example, consider the case where $n^{\alpha-\delta} \hat{\zeta}_n$ has limit distribution Q for some known value $\delta > 0$. Then, subsampling can be applied to the self-normalized quantity $(\hat{\theta}_n - \theta)/\hat{\zeta}_n$ that converges to a well-defined distribution at the *known* rate n^δ . Strictly speaking, the joint convergence of $(n^\alpha(\hat{\theta}_n - \theta), n^{\alpha-\delta} \hat{\zeta}_n)$ to the pair (J, Q) is required; see e.g. Politis et al. (1999, Theorem 11.3.1) for a precise statement.

Typically, the search for a suitable $\hat{\zeta}_n$ starts with an estimated standard deviation for $\hat{\theta}_n$ —hence the connection to studentization. The self-normalization method for subsampling was first used by Hall et al. (1998) in the context of a long-range dependent data. Hall and Yao (2003) also employ self-normalization in connection with bootstrap with smaller resample size on the residuals of heavy tailed GARCH processes. As is well-known, the bootstrap with smaller resample size is closely related to subsampling in the i.i.d. case; see Politis et al. (1999, Ch. 2.3). The case of self-normalized subsampling for heavy tailed time series was addressed by McElroy and Politis (2002), and Kokoszka and Wolf (2004). McElroy and Politis (2006) is the only paper to-date that achieves a self-normalization in a setting exhibiting both heavy-tails *and* long-range dependence.

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