Which of the following is a subspace of \( \mathbb{R}^2 \) under standard addition and standard scalar multiplication?

A. The set of all points on \( y = -3x \)
B. The set of all points on \( y = -3x + 1 \)
C. The set of all points on \( y = |x| \)
D. The set of all points on \( y = x^2 \)
E. \( \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \)
Which of the following is a subspace of $\mathbb{R}^3$ under standard addition and scalar multiplication?

A. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

B. The set of all points on the line $y = 0$

C. The set of all points on the $xz$-plane

D. The entire $\mathbb{R}^3$

E. All of the above
Question 3

Suppose two matrices $A$ and $B$ are row-equivalent. Which of the following is false?

A. $\text{Nul}(A) = \text{Nul}(B)$
B. $\text{Row}(A) = \text{Row}(B)$
C. $\text{Col}(A) = \text{Col}(B)$
D. The equations $Ax = 0$ and $Bx = 0$ have the same solution set
E. Choose this if all the above are true
Recall from Section 1.7:

**Definition**

Let \( \{v_1, v_2, \ldots, v_p\} \) be a set of \( p \) vectors in \( \mathbb{R}^n \).

- \( \{v_1, v_2, \ldots, v_p\} \) is said to be **linearly independent** if the vector equation
  \[
  x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0
  \]
  has **only** the trivial solution.

- \( \{v_1, v_2, \ldots, v_p\} \) is said to be **linearly dependent** if there exists the weights \( c_1, c_2, \ldots, c_p \), not all zeros, such that
  \[
  c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0
  \]
Definition

A set of vectors $S = \{v_1, v_2, \ldots, v_p\}$ in the vector space $V$ is called a **basis** for $V$ if

1. $S$ is linearly independent
2. $V = \text{Span}\{S\}$
Example (Standard Bases)

The following sets are **standard bases** for their respective vector spaces:

- The set $S_n = \{e_1, e_2, \ldots, e_n\}$, where $e_i$ is the $i^{th}$ column of the identity matrix $I_n$, is a basis for $\mathbb{R}^n$

- $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a basis for $M_2$

- $\{1, t, t^2, \ldots, t^n\}$ is a basis for $\mathbb{P}_n$. 
Example (Non-Standard Basis for $\mathbb{R}^3$)

The set $S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

**YOU** check:
1. The vectors are linearly independent
2. The vectors span $\mathbb{R}^3$
Let $S$ be a basis of $\mathbb{R}^3$. Then it is possible that $S$ can have more than 3 vectors

A. True. $S$ just needs to be linearly independent and span $\mathbb{R}^3$
B. False. Such a set will never span $\mathbb{R}^3$
C. False. Such a set will never be linearly independent
Let $S$ be a **basis** of $\mathbb{R}^3$. Then it is possible that $S$ can have **less than 3 vectors**

A. True. $S$ just needs to be linearly independent and span $\mathbb{R}^3$

B. False. Such a set will never span $\mathbb{R}^3$

C. False. Such a set will never be linearly independent
Remarks: Suppose we have a set of $p$ vectors $S = \{v_1, v_2, \ldots, v_p\}$ in $\mathbb{R}^n$. Then

- If $p < n$, then there is not enough vectors in $S$ to span $\mathbb{R}^n$. So $S$ is not a basis for $\mathbb{R}^n$.
- If $p > n$ then $S$ is not linearly independent. Therefore, $S$ is not a basis for $\mathbb{R}^n$. However, it’s still possible that $S$ spans $\mathbb{R}^n$.
- If $p = n$ then $S$ is a basis for $\mathbb{R}^n$ if and only if the matrix $A = [v_1 \ v_2 \ \ldots \ v_n]$ is invertible (has a pivot in every column.)
Definition

If a vector space $V$ has a basis consisting of $n$ vectors then the number $n$ is called the **dimension** of $V$, denote $dim(V) = n$.

If $V$ consists only the zero vector (i.e. $V = \{0\}$) then we define $dim(V) = 0$.

If $V$ is spanned by a finite set then $V$ is said to be **finite-dimensional**. Otherwise, $V$ is **infinite-dimensional**.
The Spanning Set Theorem

**Theorem**

Let \( S = \{v_1, v_2, \ldots, v_p\} \) be a set in \( V \) and let \( H = \text{Span}(S) \).

- If \( H \neq \{0\} \) then some subset of \( S \) is a basis of \( H \).
- If one of the vectors in \( S \) - say \( v_k \) - is the linear combination of the remaining vectors in \( S \) then the set \( S - \{v_k\} \) still spans \( H \).

**Remark:** This result allows us to obtain a basis for a vector space from it spanning set simply by removing all the “redundant” vectors in the spanning set.