Let \( B = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \) and \( C = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix} \right\} \) two non-standard bases of \( \mathbb{R}^2 \). Let \( P \) be the change-of-coordinate matrix from \( B \) to \( C \). Then \( P \) can be obtained from row-reducing which of the following augmented matrix?

A. \[
\begin{bmatrix}
2 & 1 & | & 3 & 2 \\
0 & 1 & | & 1 & -4
\end{bmatrix} \rightarrow \begin{bmatrix} I & | & P \end{bmatrix}
\]

B. \[
\begin{bmatrix}
3 & 2 & | & 2 & 1 \\
1 & -4 & | & 0 & 1
\end{bmatrix} \rightarrow \begin{bmatrix} I & | & P \end{bmatrix}
\]

C. \[
\begin{bmatrix}
2 & 0 & | & 3 & 1 \\
1 & 1 & | & 2 & -4
\end{bmatrix} \rightarrow \begin{bmatrix} I & | & P \end{bmatrix}
\]

D. \[
\begin{bmatrix}
3 & 1 & | & 2 & 1 \\
2 & 0 & | & 1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix} I & | & P \end{bmatrix}
\]
If $A$ is a $3 \times 3$ matrix such that $\det(A) = 1$ then $\det(2A)$ is

A. $\det(2A) = 2$
B. $\det(2A) = 3$
C. $\det(2A) = 4$
D. $\det(2A) = 6$
E. $\det(2A) = 8$
Which of the following is true about the determinant?

A. \( \det(A \pm B) = \det(A) \pm \det(B) \)

B. If \( A \) and \( B \) are row-equivalent then \( \det(A) = \det(B) \)

C. If \( A \) has two equal columns then \( \det(A) = 0 \)

D. If The diagonal of \( A \) has a zero entry then \( \det(A) = 0 \)

E. None of the above
1. $\det(I) = 1$.
2. Row interchange reverses the sign of the determinant.
3. The determinant is a linear function in each row separately, given that the other rows stay the same.
4. If $A$ has two equal rows then $\det(A) = 0$.
5. Adding a multiple of one row into another row ($R_i \rightarrow R_i + kR_j$) does not change the determinant.
6. If $A$ has a zero row then $\det(A) = 0$.
7. If $A$ is a triangular matrix then the determinant of $A$ is the product of the diagonal entries.
Compute the Determinant

Example

Compute $\text{det}(A)$ where

$$A = \begin{bmatrix}
0 & 1 & -3 \\
2 & -3 & 10 \\
1 & 2 & -2
\end{bmatrix}$$

Solution:

\[
\begin{vmatrix}
0 & 1 & -3 \\
2 & -3 & 10 \\
1 & 2 & -2
\end{vmatrix}
= - \begin{vmatrix}
1 & 2 & -2 \\
2 & -3 & 10 \\
0 & 1 & -3
\end{vmatrix}
= - \begin{vmatrix}
1 & 2 & -2 \\
0 & 1 & -3 \\
0 & 1 & -3
\end{vmatrix}
= -(-7) \begin{vmatrix}
1 & 2 & -2 \\
0 & 1 & -2 \\
0 & 1 & -3
\end{vmatrix}
= 7 \begin{vmatrix}
1 & 2 & -2 \\
0 & 1 & -2 \\
0 & 0 & -1
\end{vmatrix}
= 7(1)(1)(-1) = -7
\]
8. \( \det(A^T) = \det(A) \)

9. If \( A, B \) are square matrices of the same dimensions then \( \det(AB) = \det(A) \det(B) \)

10. If \( A \) is invertible then \( \det(A^{-1}) = \frac{1}{\det(A)} \)
Definition

If $A$ is a square matrix, then the **minor** $M_{ij}$ of the element $a_{ij}$ is the **determinant** of the matrix obtained by deleting the $i^{th}$ row and $j^{th}$ columns of $A$.

The **cofactor** $C_{ij}$ is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$
Recursion Formula for the Determinant - Minor and Cofactor

Example

Let

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]

Find the minor and cofactor of \( a_{21} \) and \( a_{22} \)

Answer:

a. \( M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{32}a_{13} \) and \( C_{21} = (-1)^{2+1}M_{21} = -M_{21} \)

a. \( M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{31}a_{13} \) and \( C_{22} = (-1)^{2+2}M_{21} = M_{22} \)
We can also compute the determinant through a recursion formula:

**Definition**

The determinant of square matrix $A$ is the sum of the entries in the first row of $A$ multiplied by their cofactors. That is,

$$\det(A) = \sum_{k=1}^{n} a_{1k} C_{1k} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}$$

**Your task:** Confirm this recursion formula for the $2 \times 2$ case!
Recursion Formula for the Determinant

In practice, the determinant can be evaluated by expansion along any row or column.

**Definition**

The determinant of square matrix \( A \) is given by

\[
\text{det}(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}
\]

\[
= \sum_{i=1}^{n} a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}
\]

**Comment:** This method is very useful in computing the determinant when the rows/columns contain mostly zeros.

The row/column containing the most zeros is the best choice for expansion.
Example

Find the determinant of \( A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \)

Solution:

\[
\det(A) = 3C_{13} = 3(-1)^{1+3} \left| \begin{array}{ccc} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{array} \right| \\
= 3 \left( (2)(-1)^{2+2} \left| \begin{array}{cc} -1 & 2 \\ 3 & -2 \end{array} \right| + (3)(-1)^{2+3} \left| \begin{array}{cc} -1 & 1 \\ 3 & 4 \end{array} \right| \right) \\
= 3 \left( (2)(1)(-4) + (3)(-1)(-7) \right) \\
= 39
\]