Eigenvalues & Eigenvectors - Definition

**Definition**

Let $A$ be an $n \times n$ matrix. If there exists a non-zero vector $x$ such that $Ax = \lambda x$ for some scalar $\lambda$ then

- The scalar $\lambda$ is called an **eigenvalue** of $A$
- The vector $x$ is called an **eigenvector** of $A$ corresponding to $\lambda$.

Two main questions for today:

1. Is a given non-zero vector $x$ an eigenvector of a matrix $A$? If so, find the corresponding eigenvalue.
2. Is a given scalar $\lambda$ an eigenvalue of $A$? If so, find the corresponding eigenvector(s).
Example

Let \( A = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \). Verify that \( x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( x_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \) are eigenvectors of \( A \).

Answer: Just compute \( Ax_i \) and see if this gives a scalar multiplication of \( x_i \).

\[
Ax_1 = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

\[
Ax_2 = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = (-3) \begin{bmatrix} -1 \\ 2 \end{bmatrix}
\]
Example

Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Verify that $\lambda = 5$ is an eigenvalue of $A$.

Answer: To show that $\lambda = 5$ is an eigenvalue of $A$, we need to show that the equation $Ax = 5x$ has a non-zero solution. This solution $x$ will be a corresponding eigenvector to $\lambda = 5$.

$$Ax = 5x \iff (A - 5/2)x = 0$$

Thus, $(A - 5/2)x = 0$ must have non-trivial solution.

$(A - 5/2)x = 0$ has non-trivial solution

$\iff (A - 5/2)$ is singular/non-invertible

$\iff \det(A - 5/2) = 0$
Example

Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. Find the eigenvector of $A$ corresponding to $\lambda = 5$.

Answer: Solve $(A - 5I_2)x = 0$ for non-trivial solution. Here

$$A - 5I_2 = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$ 

Solutions: $x = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ corresponding to eval 5.

Remark: This also shows that eigenvector corresponding to an eigenvalue is not unique. In this example, any nonzero scalar multiplication of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ can be used as an eigenvector.
Eigenspace

**Definition**

If $A$ is an $n \times n$ matrix and $\lambda$ is an eigenvalue of $A$. Then the set of all linear combinations of the eigenvector of $A$ corresponding to $\lambda$ is called the **eigenspace** of $A$ corresponding to $\lambda$.

**Remark:**

- The eigenspace of $A$ corresponding to $\lambda$ is the null space of $A - \lambda I$.
- While 0 cannot be an eigenvector, it is included in every eigenspace.
Properties of Eigenvalue/Eigenvector

- 0 is an eigenvalue of $A$ if and only if $A$ is non-invertible/singular. **WHY?**
- The eigenvalues of a **triangular** matrix are the entries on its diagonal.
- If $v_1 \ldots v_r$ are eigenvectors that correspond to **distinct** eigenvalues then $\{v_1 \ldots v_r\}$ are linearly independent.